ARE $\mathfrak{a}$ AND $\mathfrak{d}$ YOUR CUP OF TEA?

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Abstract. We show that consistently, every MAD family has cardinality strictly bigger than the dominating number, that is $\mathfrak{a} > \mathfrak{d}$, thus solving one of the oldest problems on cardinal invariants of the continuum. The method is a contribution to the theory of iterated forcing for making the continuum large.

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Content

§0 Introduction

We deal with the theory of iteration of forcing notions for the continuum and prove CON($\mathfrak{a} > \mathfrak{d}$) and related results. We present it in several perspectives; so §2 + §3 does not depend on §1; and §4 does not depend on §1, §2, §3. In §2 we introduce and investigate iterations which are of finite support but with non transitive memory and linear, non well ordered length and prove CON($\mathfrak{a} > \mathfrak{d}$) using a measurable. In §4 we answer also related questions ($\mathfrak{u} < \mathfrak{a}, \mathfrak{i} < \mathfrak{a}$); in §3, relying on §2 we eliminate the use of a measurable, and in §1 we rely heavily on [Sh 630].

Very basically, the difference between $\mathfrak{a}$ and $\mathfrak{b}, \mathfrak{d}$ which we use is that $\mathfrak{a}$ speaks on a set, whereas $\mathfrak{b}$ is witnessed by a sequence and $\mathfrak{d}$ by a quite directed family; it essentially deals with cofinality; so every unbounded subsequence is a witness as well, i.e. the relevant relation is transitive; when $\mathfrak{b} = \mathfrak{d}$ things are smooth, otherwise the situation is still similar. This manifests itself by using ultrapowers for some κ-complete ultrafilter (in model theoretic outlook), and by using "convergent sequence" (see [Sh 300], or the existence of Av, the average, in [Sh:c]) in §2, §3, respectively. The meaning of "model theoretic outlook", is that by experience set theorists starting to hear an explanation of the forcing tend to think of an elementary embedding $\mathbf{j} : \mathbf{V} \rightarrow M$ and then the limit practically does not make sense (though of course we can translate). Note that ultrapowers by e.g. an ultrafilter on κ, preserve any witness for a cofinality of a linear order being $\geq \kappa^+$ (or the cofinality of a κ^+-directed partial order), as the set of old elements is cofinal and a cofinal subset of a cofinal subset is a cofinal subset. On the other hand, the ultrapower always "increase" a set of cardinality at least the completeness of the ultrafilter.

* * *

This is one of the oldest problems on cardinal invariants of the continuum (see [vD]). It was mostly thought that consistently $\mathfrak{a} > \mathfrak{d}$ and that the natural way to proceed is by CS iteration $\langle P_i, \underset{\sim}{Q}_i : i < \omega_2 \rangle$ of proper ${}^\omega\omega$-bounding forcing notions, starting with $\mathbf{V} \models$ GCH, and $|P_i| = \aleph_1$ for $i < \omega_2$ and $\underset{\sim}{Q}_i$ "deal" with one MAD family $\mathscr{A}_i \in V^{P_i}, \mathscr{A}_i \subseteq [\omega]^{\aleph_0}$, adding an infinite subset of ω almost disjoint to every $A \in \mathscr{A}_i$. The needed iteration theorem holds by [Sh:f, Ch.V,§4], saying that in $\mathbf{V}^{P_{\omega_2}}, \mathfrak{d} = \mathfrak{b} = \aleph_1$ and no cardinal is collapsed, <u>but</u> the single step forcing is not known to exist. This has been explained in details in [Sh 666].

We do not go in this way but in a totally different direction involving making the continuum large, so we still do not know

<u>0.1 Problem</u> Is $ZFC + 2^{\aleph_0} + \aleph_2 + \mathfrak{a} > \mathfrak{d}$ consistent?

To clarify our idea, let D be a normal ultrafilter on κ, a measurable cardinal and consider a c.c.c. forcing notion P and

(a) a sequence $\bar{\underset{\sim}{f}} = \langle \underset{\sim}{f}_\alpha : \alpha < \kappa^+\rangle$ of P-names such that
$\Vdash_P$ "$\langle \underset{\sim}{f}_\alpha : \alpha < \kappa^+\rangle$ is $<^*$-increasing cofinal in ${}^\omega\omega$"
(so $\bar{\underset{\sim}{f}}$ exemplifies $\Vdash_P$ "$\mathfrak{b} = \mathfrak{d} = \kappa^+$")

(b) a sequence $\langle \underset{\sim}{A}_\alpha : \alpha < \alpha^*\rangle$ of P-names such that
$\Vdash_P$ "$\{\underset{\sim}{A}_\alpha : \alpha < \alpha^*\}$ is MAD that is $\alpha \neq \beta \Rightarrow \underset{\sim}{A}_\alpha \cap \underset{\sim}{A}_\beta$ is finite and $\underset{\sim}{A}_\alpha \in [\omega]^{\aleph_0}$".

Now $P_1 = P^\kappa/D$ also is a c.c.c. forcing notion by Łoś theorem for $L_{\kappa,\kappa}$; let $\mathbf{j} : P \to P_1$ be the canonical embedding; moreover, under the canonical identification we have $P \prec_{L_{\kappa,\kappa}} P_1$. So also $\Vdash_{P_1}$ "$\underset{\sim}{f}_\alpha \in {}^\omega\omega$", recalling that $\underset{\sim}{f}_\alpha$ actually consists of ω maximal antichains of P (or think of $(\mathscr{H}(\chi), \in)^\kappa/D, \chi$ large enough). Similarly $\Vdash_{P_1}$ "$\underset{\sim}{f}_\alpha <^* \underset{\sim}{f}_\beta$ if $\alpha < \beta < \kappa^+$".

Now, if $\Vdash_{P_1}$ "$\underset{\sim}{g} \in {}^\omega\omega$", then $\underset{\sim}{g} = \langle \underset{\sim}{g}_\varepsilon : \varepsilon < \kappa\rangle/D, \Vdash_P$ "$\underset{\sim}{g}_\varepsilon \in {}^\omega\omega$" so for some $\alpha^* < \kappa^+$ we have $\Vdash_P$ "$\underset{\sim}{g}_\varepsilon <^* \underset{\sim}{f}_\alpha$ for $\varepsilon < \kappa$" hence by Łoś theorem $\Vdash_{P_1}$ "$\underset{\sim}{g} <^* f_\alpha$" (so before the identification this means $\Vdash_{P_1}$ "$\underset{\sim}{g} <^* \mathbf{j}(f_\alpha)$"), so $\langle \underset{\sim}{f}_\alpha : \alpha < \kappa^+\rangle$ exemplifies also $\Vdash_{P_1}$ "$\mathfrak{b} = \mathfrak{d} = \kappa^+$".

On the other hand $\langle \underset{\sim}{A}_\alpha : \alpha < \alpha^*\rangle$ cannot exemplify that $\mathfrak{a} \leq \kappa^+$ in $\mathbf{V}^{P_1}$ because $\alpha^* \geq \kappa^+$ (as $ZFC \models \mathfrak{b} \leq \mathfrak{a}$) so $\langle \underset{\sim}{A}_\alpha : \alpha < \kappa\rangle/D$ exemplifies that $\Vdash_{P_1}$ "$\{\underset{\sim}{A}_\alpha : \alpha < \alpha^*\}$ is not MAD".

Our original idea here is to start with a FS iteration $\bar{Q}^0 = \langle P^0_i, \underset{\sim}{Q}^0_i : i < \kappa^+\rangle$ of nep c.c.c. forcing notions, $\underset{\sim}{Q}^0_i$ adding a dominating real, (e.g. dominating real = Hechler forcing), for κ a measurable cardinal and let D be a κ-complete uniform ultrafilter on κ and $\chi >> \kappa$. Then let $L_0 = \kappa^+, \bar{Q}^1 = \langle P^1_i, Q^1_i : i \in L_1\rangle$ be $\bar{Q}^0$ as interpreted in $(\mathscr{H}(\chi), \in, <^*_\chi)^\kappa/D$, it looks like $\bar{Q}^0$ replacing κ^+ by $(\kappa^+)^\kappa/D$. We look at $\mathrm{Lim}(\bar{Q}^0) = \bigcup\limits_i P_i$ as a subforcing of $\mathrm{Lim}(\bar{Q}^1)$ identifying $\underset{\sim}{Q}_i$ with $\underset{\sim}{Q}_{\mathbf{j}_0(i)}, \mathbf{j}_0$ the canonical elementary embedding of κ^+ into $(\kappa^+)^\kappa/D$ (no Mostowski collapse!). We continue to define $\bar{Q}^n$ and then $\bar{Q}^\omega$ as the following limit: for the original $i \in \kappa^+$, we use the definition, otherwise we use direct limit ("founding fathers priviledge" you may say). So $P^i =$ $\mathrm{Lim}(\bar{Q}^i)$ is $\lessdot$-increasing, continuous when $\mathrm{cf}(i) > \aleph_0$; so now we have a kind of iteration with non transitive memory and not well founded.

We continue κ^{++} times. Now in $\mathbf{V}^{\mathrm{Lim}(\bar{Q}^{\kappa^{++}})}$, the original κ^+ generic reals exemplify $\mathfrak{b} = \mathfrak{d} = \kappa^+$, so we know that $\mathfrak{a} \geq \kappa^+$. To finish assume $p \Vdash$ "$\{\underset{\sim}{A}_\gamma : \gamma < \kappa^+\} \subseteq [\omega]^{\aleph_0}$ is a MAD family". Each name $\underset{\sim}{A}_\gamma$ is a "countable object" and so depends on countably many co-ordinates, so all of them are in $\mathrm{Lim}(\bar{Q}^i)$ for some $i < \kappa^{++}$. In the next stage, $\bar{Q}^{i+1}$, $\langle \underset{\sim}{A}_\gamma : \gamma < \kappa \rangle / D$ is a name of an infinite subset of ω almost disjoint to $\underset{\sim}{A}_\beta$ for each $\beta < \kappa^+$, contradiction.

All this is a reasonable scheme. This is done in §1 but relay on "nep forcing" from [Sh 630]. But a self contained another approach in §2,§3, where the meaning of the iteration is more on the surface (and also, in §3, help to eliminate the use of large cardinals). In §4 we deal with the case of an additional cardinal invariant, $\mathfrak{u}$.

Note that just using FS iteration on a non well-ordered linear order L (instead of an ordinal) is impossible by a theorem of Hjorth. On nonlinear orders for iterations (history and background) see [RoSh 670]. On iteration with nontransitive memory see [Sh 592], [Sh 619] and in particular [Sh 619, §3].

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§1 On Con($\mathfrak{a} > \mathfrak{d}$)

In this section, we look at it in the context of [Sh 630] and we use a measurable.

1.1 Definition. 1) Given sets A_ℓ of ordinals for $\ell < n$, we say $\mathscr{T}$ is an $(A_0, \ldots, A_{n-1})$-tree if $\mathscr{T} = \bigcup_{k<\omega} \mathscr{T}_k$ where $\mathscr{T}_k \subseteq \{(\eta_0, \ldots, \eta_\ell, \ldots, \eta_{n-1}) : \eta_\ell \in {}^k(A_\ell) \text{ for } \ell < n\}$ and $\mathscr{T}$ is ordered by $\bar\eta \le_{\mathscr{T}} \bar\nu \Leftrightarrow \bigwedge_{\ell<n} \eta_\ell \trianglelefteq \nu_\ell$ and we let $\bar\eta \restriction k_1 =: \langle \eta_\ell \restriction k_1 : \ell < n\rangle$ and demand $\bar\eta \in \mathscr{T}_k \;\&\; k_1 < k \Rightarrow \bar\eta \restriction k_1 \in \mathscr{T}_{k_1}$. We call $\mathscr{T}$ locally countable if $k \in [1,\omega) \;\&\; \bar\eta \in \mathscr{T}_k \Rightarrow |\{\bar\nu \in \mathscr{T}_{k+1} : \bar\eta \le_{\mathscr{T}} \bar\nu\}| \le \aleph_0$. Let $\lim(\mathscr{T}) = \{\langle \eta_\ell : \ell < n\rangle : \eta_\ell \in {}^\omega(A_\ell)$ for $\ell < k$ and $m < \omega \Rightarrow \langle \eta_\ell \restriction m : \ell < n\rangle \in \mathscr{T}\}$. Lastly for $n_1 \le n$ we let $\text{prj}\lim_{n_1}(\mathscr{T}) = \{\langle \eta_\ell : \ell < n_1\rangle :$ for some $\eta_{n_1}, \ldots, \eta_{n-1}$ we have $\langle \eta_\ell : \ell < n\rangle \in \lim(\mathscr{T})\}$; and if n_1 is omitted we mean $n_1 = n-1$.
2)

$\mathfrak{K} = \big\{ \bar{\mathscr{T}}$:for some sets A, B of ordinals we have

(i) $\bar{\mathscr{T}} = (\mathscr{T}_1, \mathscr{T}_2)$,

(ii) $\mathscr{T}_1$ is a locally countable (A,B)-tree,

(iii) $\mathscr{T}_2$ is a locally countable (A,A,B)-tree, and

(iv) $Q_{\bar{\mathscr{T}}} =: (\text{prj}\lim(\mathscr{T}_1),\ \text{prj}\lim(\mathscr{T}_2))$ is a c.c.c. forcing notion absolutely under c.c.c. forcing notions (see below)$\big\}$

2A) We say that $Q_{\bar{\mathscr{T}}}$ is c.c.c. absolutely for c.c.c. forcing if: for c.c.c. forcing notions $P \lessdot R$ we have $Q^{\mathbf{V}^P}_{\bar{\mathscr{T}}} \lessdot Q^{\mathbf{V}^R}_{\bar{\mathscr{T}}}$ so membership, order, nonorder, compatibility, noncompatibility and being predense over p are preserved (the $Q_{\bar{\mathscr{T}}}$'s are snep, from [Sh 630] with slight restriction). Similarly we define "$Q_{\bar{\mathscr{T}}} \lessdot Q_{\bar{\mathscr{T}}}$ absolutely under c.c.c. forcing".
3) For a set or class A of ordinals, $\mathfrak{K}^\kappa_A$ is the family of $\bar{\mathscr{T}} \in \mathfrak{K}$ which are a pair of objects, the first an (A,B)-tree and the second an (A,A,B)-trees for some B such that $|\mathscr{T}_1| \le \kappa, |\mathscr{T}_2| \le \kappa$. For a cardinal κ and a pairing function pr with inverses pr_1, pr_2, let $\mathfrak{K}^\kappa_{\text{pr}_1,\gamma} = \mathfrak{K}^\kappa_{\{\alpha:\text{pr}_1(\alpha)=\gamma\}}$ and $\mathfrak{K}^\kappa_{\text{pr}_1,<\gamma} = \mathfrak{K}^\kappa_{\{\alpha:\text{pr}_i(\alpha)<\gamma\}}$. Let $|\bar{\mathscr{T}}| = |\mathscr{T}_1| + |\mathscr{T}_2|$.
4) Let $\bar{\mathscr{T}}, \bar{\mathscr{T}}' \in \mathfrak{K}$, we say $\mathbf{f}$ is an isomorphism from $\bar{\mathscr{T}}$ onto $\bar{\mathscr{T}}'$ <u>when</u>: $\mathbf{f} = (f_1, f_2)$ and for $m = 1,2$ we have: f_m is a one-to-one function from $\mathscr{T}_m$ onto $\mathscr{T}'_m$ preserving the level (in the respective trees), preserving the relations $x = y \restriction k, x \ne y \restriction k$ and if $f_2((\eta_1,\eta_2,\eta_3)) = (\eta'_1,\eta'_2,\eta'_3), f_1((\nu_1,\nu_2)) = (\nu'_1,\nu'_2)$ then $[\eta_1 = \nu_1 \Leftrightarrow \eta'_1 = \nu'_1], [\eta_2 = \nu_1 \Leftrightarrow \eta'_2 = \nu'_1]$.
In this case let $\hat{\mathbf{f}}$ be the isomorphism induced by $\mathbf{f}$ from $Q_{\bar{\mathscr{T}}}$ onto $Q_{\bar{\mathscr{T}}'}$.

1.2 Definition. For $\bar{\mathscr{T}}', \bar{\mathscr{T}}'' \in \mathfrak{K}$ let $\bar{\mathscr{T}}' \leq_{\mathfrak{K}} \bar{\mathscr{T}}''$ mean:

(a) $\mathscr{T}'_\ell \subseteq \mathscr{T}''_\ell$ (as trees) for $\ell = 1, 2$

(b) if $\ell \in \{1, 2\}$ and $\bar{\eta} \in \mathscr{T}''_\ell \backslash \mathscr{T}'_\ell$ and $\bar{\eta} \restriction k \in \mathscr{T}'_\ell$ <u>then</u> $k \leq 1$

(c) $Q_{\bar{\mathscr{T}}'} \lessdot Q_{\bar{\mathscr{T}}''}$ (absolutely under c.c.c. forcing); note that by (a) + (b) we have:
$x \in Q_{\bar{\mathscr{T}}'} \Rightarrow x \in Q_{\bar{\mathscr{T}}''}$ and $Q_{\bar{\mathscr{T}}'} \models x \leq y \Rightarrow Q_{\bar{\mathscr{T}}''} \models x \leq y$).

Remark. The definition is tailored such that the union of an increasing chain will give a forcing notion which is the union.

1.3 Claim/Definition. *0) $\leq_{\mathfrak{K}}$ is a partial order of $\mathfrak{K}$.*
1) Assume $\langle \bar{\mathscr{T}}[i] : i < \delta \rangle$ is $\leq_{\mathfrak{K}}$-increasing and $\bar{\mathscr{T}}$ is defined by $\bar{\mathscr{T}} = \bigcup\limits_i \bar{\mathscr{T}}[i]$ that is $\mathscr{T}_m = \bigcup\limits_{i<\delta} \mathscr{T}_m[i]$ for $m = 1, 2$ <u>then</u>

(a) $i < \delta \Rightarrow \bar{\mathscr{T}}[i] \leq_{\mathfrak{K}} \bar{\mathscr{T}}$

(b) $Q_{\bar{\mathscr{T}}} = \bigcup\limits_{i<\delta} Q_{\bar{\mathscr{T}}[i]}$.

2) Assume $\bar{\mathscr{T}}', \bar{\mathscr{T}} \in \mathfrak{K}$. <u>Then</u> there is $\bar{\mathscr{T}}'' \in \mathfrak{K}$ such that $\bar{\mathscr{T}}' \leq_{\mathfrak{K}} \bar{\mathscr{T}}''$ and $Q_{\bar{\mathscr{T}}''}$ is isomorphic to $Q_{\bar{\mathscr{T}}'} * \underset{\sim}{Q}_{\bar{\mathscr{T}}}$ and this is absolute by c.c.c. forcing. Moreover, there is such an isomorphism extending the identity map from $Q_{\bar{\mathscr{T}}'}$ into $Q_{\bar{\mathscr{T}}''}$.
3) There is $\bar{\mathscr{T}} \in \mathfrak{K}^{\aleph_0}_\omega$ such that $Q_{\bar{\mathscr{T}}}$ is the trivial forcing.
4) There is $\bar{\mathscr{T}} \in \mathfrak{K}^{\aleph_0}_\omega$ such that $Q_{\bar{\mathscr{T}}}$ is the dominating real forcing.

Proof. See [Sh 630].

1.4 Claim. *1) Assume $\bar{\mathscr{T}}[\gamma] \in \mathfrak{K}_{\mathrm{pr}_1,\gamma}$ for $\gamma < \gamma(*)$. <u>Then</u> for each $\alpha \leq \gamma(*)$ there is $\bar{\mathscr{T}}\langle\alpha\rangle \in \mathfrak{K}_{\mathrm{pr}_1,\gamma<\gamma(*)}$ such that $Q_{\bar{\mathscr{T}}\langle\alpha\rangle}$ is P_α where $\langle P_\gamma, \underset{\sim}{Q}_\beta : \gamma \leq \gamma(*), \beta < \gamma(*) \rangle$ is an FS iteration and $\underset{\sim}{Q}_\beta = (Q_{\bar{\mathscr{T}}[\beta]})^{\mathbf{V}[P_\beta]}$ and $\bar{\mathscr{T}}\langle\alpha\rangle \in \mathfrak{K}_{\mathrm{pr}_1,<\alpha}$ and $\bar{\mathscr{T}}\langle\alpha_1\rangle \leq_{\mathfrak{K}} \bar{\mathscr{T}}\langle\alpha_2\rangle$ for $\alpha_1 \leq \alpha_2 \leq \gamma(*)$, $\bar{\mathscr{T}}[\gamma] \leq_{\mathfrak{K}} \bar{\mathscr{T}}\langle\alpha\rangle$ for $\gamma < \alpha \leq \gamma(*)$. We write $\bar{\mathscr{T}}\langle\alpha\rangle = \sum\limits_{\gamma<\alpha} \bar{\mathscr{T}}[\gamma]$.*
2) In part (1), for each $\gamma < \gamma(*)$ there is $\bar{\mathscr{T}}' \in \mathfrak{K}_{\mathrm{pr}_1,\gamma}$ such that $\bar{\mathscr{T}}', \bar{\mathscr{T}}$ are isomorphic over $\bar{\mathscr{T}}[\gamma]$ hence $Q_{\bar{\mathscr{T}}'}, Q_{\bar{\mathscr{T}}}$ are isomorphic over $Q_{\bar{\mathscr{T}}[\gamma]}$.
3) If in addition $\mathscr{T}[\gamma] \leq_{\mathfrak{K}} \mathscr{T}'[\gamma] \in \mathfrak{K}_{\mathrm{pr}_1,\gamma}$ for $\gamma < \gamma(*)$ and $\langle P_\gamma, \underset{\sim}{Q}'_\beta : \gamma \leq \gamma(*),$

$\beta < \gamma(*)\rangle$ is an FS iteration as above with $P'_{\gamma(*)} = Q_{\bar{\mathscr{T}}'}$, <u>then</u> we find such $\bar{\mathscr{T}}'$ with $\bar{\mathscr{T}} \leq_{\mathfrak{K}} \bar{\mathscr{T}}'$.

Proof. Straight.

1.5 Theorem. *Assume*

- (a) *κ is a measurable cardinal*
- (b) *$\kappa < \mu = cf(\mu) < \lambda = cf(\lambda) = \lambda^\kappa$ and $(\forall \alpha < \mu)(|\alpha|^{\aleph_0} < \mu)$ for simplicity.*

<u>Then</u> for some c.c.c. forcing notion P of cardinality λ, in $\mathbf{V}^P$ we have: $2^{\aleph_0} = \lambda, \mathfrak{d} = \mathfrak{b} = \mu$ and $\mathfrak{a} = \lambda$.

Proof. We choose by induction on $\zeta \leq \lambda$ the following objects satisfying the following conditions:

- (a) a sequence $\langle \bar{\mathscr{T}}[\gamma,\zeta] : \gamma < \mu\rangle$
- (b) $\bar{\mathscr{T}}[\gamma,\zeta] \in \mathfrak{K}^\lambda_{\mathrm{pr}_1,\gamma}$
- (c) $\xi < \zeta \Rightarrow \bar{\mathscr{T}}[\gamma,\xi] \leq_{\mathfrak{K}} \bar{\mathscr{T}}[\gamma,\zeta]$
- (d) if ζ limit then $\bar{\mathscr{T}}[\gamma,\zeta] = \bigcup\limits_{\xi<\zeta} \bar{\mathscr{T}}[\gamma,\xi]$
- (e) if $\gamma < \mu, \zeta = 1$ <u>then</u> $Q_{\bar{\mathscr{T}}}[\gamma,\zeta]$ is the dominating real forcing = Hechler forcing
- (f) if $\gamma < \mu, \zeta = \xi + 1 > 1$ and ξ is even, <u>then</u> $\bar{\mathscr{T}}[\gamma,\zeta]$ is isomorphic to $\bar{\mathscr{T}}\langle\gamma+1,\xi\rangle$ over $\bar{\mathscr{T}}[\gamma,\xi]$ say by $\mathbf{j}_{\gamma,\xi}$ where $\bar{\mathscr{T}}\langle\gamma+1,\xi\rangle =: \sum\limits_{\beta\leq\gamma} \bar{\mathscr{T}}[\beta,\xi]$ and let $\hat{\mathbf{j}}_{\gamma,\xi}$ be the isomorphism induced from $Q_{\bar{\mathscr{T}}\langle\gamma+1,\xi\rangle}$ onto $Q_{\bar{\mathscr{T}}}[\gamma,\zeta]$ over $Q_{\bar{\mathscr{T}}[\gamma,\xi]}$
- (g) if $\gamma < \mu, \zeta = \xi + 1, \xi$ odd, <u>then</u> $\bar{\mathscr{T}}[\gamma,\zeta]$ is almost isomorphic to $(\bar{\mathscr{T}}[\gamma,\xi])^\kappa/D$ over $\bar{\mathscr{T}}_{[\gamma,\xi]}$ say $\mathbf{j}_{\gamma,\xi}$ is an almost isomorphism from $(\bar{\mathscr{T}}[\gamma,\xi])^\kappa/D$ onto $\bar{\mathscr{T}}[\gamma,\zeta]$ such that by $\mathbf{j}_{\gamma,\xi}$ $\langle x : \varepsilon < \kappa\rangle/D$ is mapped onto x.

There is no problem to carry the definition. Let $P_\zeta = Q_{\bar{\mathscr{T}}\langle\mu,\zeta\rangle}$ where $\bar{\mathscr{T}}\langle\mu,\zeta\rangle =: \sum\limits_{\gamma<\mu} \bar{\mathscr{T}}[\gamma,\zeta]$ for $\zeta \leq \lambda, P = P_\lambda$ and $P_{\gamma,\zeta} = Q_{\bar{\mathscr{T}}\langle\gamma,\zeta\rangle}$. Now

$\boxtimes_1$ $|P| \leq \lambda$
[why? as we prove by induction on $\zeta \leq \lambda$ that: each $\bar{\mathscr{T}}[\gamma,\zeta]$ and $\sum\limits_{\gamma\leq\mu} \bar{\mathscr{T}}[\gamma,\lambda]$

has cardinality $\le \lambda$. Hence for $\gamma < \mu$ we have: the forcing notion $Q_{\bar{\mathscr{T}}[\gamma,\lambda]}$ in the universe $\mathbf{V}^{Q_{\bar{\mathscr{T}}\langle\gamma,\lambda\rangle}}$ has cardinality $\le \lambda^{\aleph_0} = \lambda$]

$\boxtimes_2$ in $\mathbf{V}^P$ we have $\mathfrak{b} = \mathfrak{d} = \lambda$
[why? let $\underset{\sim}{\eta}_\gamma$ be the $Q_{\bar{\mathscr{T}}[\gamma,1]}$-name of the dominating real (see clause (e)). As $\bar{\mathscr{T}}[\gamma,1] \le_{\mathfrak{K}} \bar{\mathscr{T}}[\gamma,\lambda]$, clearly $\underset{\sim}{\eta}_\gamma$ is also a $Q_{\bar{\mathscr{T}}[\gamma,\lambda]}$-name of a dominating real, so $\Vdash_P$ "$\underset{\sim}{\eta}_\gamma$ dominate $({}^\omega\omega)^{\mathbf{V}[P_{\gamma,\lambda}]}$". But $\langle P_{\gamma,\lambda} : \gamma < \mu\rangle$ is $\lessdot$-increasing with union P and $\mathrm{cf}(\mu) = \mu > \aleph_0$ so $\Vdash_P$ "$\langle \underset{\sim}{\eta}_\gamma : \gamma < \mu\rangle$ is $<^*$-increasing and dominating", so the conclusion follows.]

We shall prove below that $\mathfrak{a} \ge \lambda$, together this finishes the proof (note that it implies $2^{\aleph_0} \ge \lambda$ hence as $\lambda = \lambda^{\aleph_0}$ by $\boxtimes_1$ we get $2^{\aleph_0} = \lambda$)

$\boxtimes_3$ $\Vdash_P$ "$\mathfrak{a} \ge \lambda$".

So assume $p \Vdash$ "$\underset{\sim}{\mathscr{A}} = \{\underset{\sim}{A}_i : i < \theta\}$ is a MAD family, i.e. ($\theta \ge \aleph_0$ and)

(*i*) $\underset{\sim}{A}_i \in [\omega]^{\aleph_0}$,

(*ii*) $i \ne j \Rightarrow |\underset{\sim}{A}_i \cap \underset{\sim}{A}_j| < \aleph_0$ and

(*iii*) $\underset{\sim}{\mathscr{A}}$ is maximal under $(i) + (ii)$".

Without loss of generality $\Vdash_P$ "$\underset{\sim}{A}_i \in [\omega]^{\aleph_0}$".

As always $\mathfrak{a} \ge \mathfrak{b}$, by $\boxtimes_2$ we know that $\theta \ge \mu$, and toward contradiction assume $\theta < \lambda$. For each $i < \theta$ and $m < \omega$ there is a maximal antichain $\langle p_{i,m,n} : n < \omega\rangle$ of P and a sequence $\langle \mathbf{t}_{i,m,n} : n < \omega\rangle$ of truth values such that $p_{i,m,n} \Vdash_P$ "$n \in \underset{\sim}{A}_i$ iff $\mathbf{t}_{i,m,n}$ is truth". We can find a countable $w_i \subseteq \mu$ such that: $\big[\bigcup\limits_{m,n<\omega} \mathrm{Dom}(p_{i,m,n}) \subseteq w_i\big]$, $p_{i,m,n} \in Q_{\cup\{\bar{\mathscr{T}}[\gamma,\lambda]:\gamma\in w_i\}}$, moreover, $\gamma \in \mathrm{Dom}(p_{i,m,n}) \Rightarrow p_{i,m,n}(\gamma)$ is a $Q_{\sum\{\bar{\mathscr{T}}[\beta,\lambda]:\beta\in\gamma\cap w_i\}}$-name. Note that $Q_{\sum\{\bar{\mathscr{T}}[\beta,\lambda]:\beta\in\gamma\cap w_i, i<\theta\}} \lessdot Q_{\sum\{\bar{\mathscr{T}}_\beta:\beta<\gamma\}}$, see [Sh 630].

Clearly for some even $\zeta < \lambda$, we have $\{p_{i,m,n} : i < \theta, m < \omega \text{ and } n < \omega\} \subseteq Q_{\sum\{\bar{\mathscr{T}}[\beta,\zeta]:\beta<\mu\}}$. Now for some stationary $S \subseteq \{\delta < \mu : \mathrm{cf}(\delta) = \kappa\}$ and w^* we have: $\delta \in S \Rightarrow w_\delta \cap \delta = w^*$ and $\alpha < \delta \in S \Rightarrow w_\alpha \subseteq \delta$. Let $\langle \delta_\varepsilon : \varepsilon < \kappa\rangle$ be an increasing sequence of members of S, and $\delta^* = \bigcup\limits_{\varepsilon<\kappa} \delta_\varepsilon$. The definition of $\langle \bar{\mathscr{T}}[\gamma,\zeta+1] : \gamma < \mu\rangle, \langle \bar{\mathscr{T}}[\gamma,\zeta+2] : \gamma < \mu\rangle$ was made to get a name of an infinite

$\underset{\sim}{A} \subseteq \omega$ almost disjoint to every $\underset{\sim}{A}_\beta$ for $\beta < \theta$ (in fact $(\sum\limits_{\gamma<\mu} Q_{\bar{\mathscr{T}}[\gamma,\zeta]})^\kappa/D$ can be $\lessdot$-embedded into $\sum\limits_{\gamma<\mu} Q_{\bar{\mathscr{T}}[\gamma,\zeta+2]}$). $\square_{1.5}$

Remark. In later proofs in §2 we give more details.

§2
On Con($\mathfrak{a} > \mathfrak{d}$) revisited with FS, with non transitive memory, non-well ordered length

We first define the FSI-templates, telling us how do we iterate along a linear order L; we think of having for each $t \in L$, a forcing notion Q_t, say adding a generic $\underset{\sim}{\nu}_t$, and Q_t will really be $\cup\{Q^{\mathbf{V}[\langle \underset{\sim}{\nu}_s : s \in A\rangle]} : A \in I_t\}$ where I_t an ideal of subsets of $\{s : s <_L t\}$; so Q_t in the nice case is a definition. In our application this definition is constant, but we treat a more general case, so $\underset{\sim}{Q}_t$ may be defined using parameters from $\mathbf{V}[\langle \underset{\sim}{\nu}_s : s \in K_t\rangle], K_t$ a subset of $\{s : s <_L t\}$ so the reader may consider only the case $t \in L \Rightarrow K_t = \emptyset$. In part (3) instead distinguishing "ζ odd, ζ even" we can consider the two cases for each ζ. The depth of L is the ordinal on which our induction rests (as $\mathrm{otp}(L)$ is inadequate).

2.1 Definition. 1) An FSI-template (= finite support iteration template) $\mathfrak{t}$ is a sequence $\langle I_t : t \in L\rangle = \langle I^{\mathfrak{t}}_t : t \in L^{\mathfrak{t}}\rangle = \langle I_t[\mathfrak{t}] : t \in L[\mathfrak{t}]\rangle$ such that

- (a) L is a linear order (but we may write $x \in \mathfrak{t}$ instead of $x \in L$ and $x <_{\mathfrak{t}} y$ instead of $x <_L y$)
- (b) I_t is an ideal of subsets of $\{s : L \models s < t\}$.

We say $\mathfrak{t}$ is locally countable if $t \in L^{\mathfrak{t}}$ & $(\forall B \in [A]^{\aleph_0})(B \in I_t) \Rightarrow A \in I_t$ and we say $\mathfrak{t}^1, \mathfrak{t}^2$ are equivalent if $L^{\mathfrak{t}^1} = L^{\mathfrak{t}^2}$ and $t \in L^{\mathfrak{t}^1}$ & $|A| \le \aleph_0 \Rightarrow (A \in I^{\mathfrak{t}^1}_t \equiv A \in I^{\mathfrak{t}^2}_t)$.
2) Let $\mathfrak{t}$ be an FSI-template.

- (c) We say $\bar{K} = \langle K_t : t \in L^{\mathfrak{t}}\rangle$ is a $\mathfrak{t}$-memory choice if
  - (i) $K_t \in I^{\mathfrak{t}}_t$ is countable
  - (ii) $s \in K_t \Rightarrow K_s \subseteq K_t$.
- (d) We say $L \subseteq L^{\mathfrak{t}}$ is $\bar{K}$-closed if $t \in L \Rightarrow K_t \subseteq L$
- (e) for $\bar{K}$ a $\mathfrak{t}$-memory choice and $L \subseteq L^{\mathfrak{t}}$ which is $\bar{K}$-closed we say $\bar{K}' = \bar{K} \restriction L$ if $\mathrm{Dom}(\bar{K}') = L$ and K'_t is K_t for $t \in L$, (it is a $(\mathfrak{t} \restriction L)$-memory choice, see part (5)).

3) For an FSI-template $\mathfrak{t}$ and $\mathfrak{t}$-memory choice $\bar{K}$ and $\bar{K}$-closed $L \subseteq L^{\mathfrak{t}}$ we define $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K})$, the $\mathfrak{t}$-depth (or $(\mathfrak{t}, \bar{K})$-depth) of L by defining by induction on the ordinal ζ when $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) \le \zeta$.

<u>For $\zeta = 0$</u>: $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) \le \zeta$ when $L = \emptyset$.

<u>For ζ odd</u>: $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) \le \zeta$ iff:

(a) there is L^* such that: $L^* \subseteq L, (\forall t \in L)(\forall A \in I^{\mathfrak{t}}_t)(A \cap L^* = \emptyset)$ hence $L \backslash L^*$ is $\bar{K}$-closed and $\mathrm{Dp}_{\mathfrak{t}}(L \backslash L^*, \bar{K}) < \zeta$ and for every $t \in L^*$ we have

$\boxtimes_{t,L}$ in $\{A \in I^{\mathfrak{t}}_t : A \subseteq L\}$ there is a maximal element[1] and it is $\bar{K}$-closed,

<u>For $\zeta > 0$ even</u>: $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) \le \zeta$ iff:

(b) there is a directed partial order M and a sequence $\langle L_a : a \in M \rangle$ with union L such that $M \models a \le b \Rightarrow L_a \subseteq L_b$, each L_b is $\bar{K}$-closed, $(\forall b \in M)(\zeta > \mathrm{Dp}_{\mathfrak{t}}(L_b, \bar{K}))$ and $t \in L \ \& \ A \in I_t \ \& \ A \subseteq L \Rightarrow (\exists a \in M)\, A \subseteq L_a$.

So $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) = \zeta$ iff $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) \ge \zeta \ \& \ (\forall \xi < \zeta)\ \mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) \nleq \xi$ and $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) = \infty$ iff ($\forall$ ordinal ζ) $[\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) \nleq \zeta]$.
4) We say $\bar{K}$ is a smooth $\mathfrak{t}$-memory choice if $\mathrm{Dp}_{\mathfrak{t}}(L^{\mathfrak{t}}, \bar{K}) < \infty$ and $\bar{K}$ a $\mathfrak{t}$-memory choice.
5) If $\bar{K}$ is omitted we mean $K_t = \emptyset$ for $t \in L^{\mathfrak{t}}$. We say $\mathfrak{t}$ is smooth if the trivial $\bar{K}$ is a smooth $\mathfrak{t}$-memory choice. For $L \subseteq L^{\mathfrak{t}}$ let $\mathfrak{t} \restriction L = \langle I_t \cap \mathscr{P}(L) : t \in L \rangle$.
6) Let $L_1 \le_{\mathfrak{t}} L_2$ mean $L_1 \subseteq L_2 \subseteq L^{\mathfrak{t}}$ and $t \in L_1 \ \& \ A \in I^{\mathfrak{t}}_t \Rightarrow A \cap L_2 \subseteq L_1$.

2.2 Definition. Let $\mathfrak{t} = \langle I_t : t \in L^{\mathfrak{t}} \rangle$ be a FS iteration template and $\bar{K}$ a $\mathfrak{t}$-memory choice.
1) We say $\bar{L}$ is a $(\mathfrak{t}, \bar{K})$-representation of L if:

(a) $L \subseteq L^{\mathfrak{t}}$ is $\bar{K}$-closed

(b) $\bar{L} = \langle L_a : a \in M \rangle$

(c) M is a directed partial order

(d) $\bar{L}$ is increasing, that is $a <_M b \Rightarrow L_a \subseteq L_b$

(e) $L = \bigcup\limits_{a \in M} L_a$

(f) each L_a is $\bar{K}$-closed

(g) if $t \in L, A \in I^{\mathfrak{t}}_t, A \subseteq L$ then $(\exists a \in M)(A \subseteq L_a)$.

[1] we can use less, it seems not needed at the moment. We can go deeper to names of depth $\le \varepsilon$ inductively on $\varepsilon < \omega_1$, as in [Sh 619, §3], or in a more particular way to make the point this is used here true, and/or make $I^{\mathfrak{t}}_t$ only closed under unions (but not subsets), etc.
Note that e.g. $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$ is well defined when $L^{\mathfrak{t}}$ is well ordered.

2) We say $(L^*, \bar{A})$ is a $(\mathfrak{t}, \bar{K})-{}^*$representation of L if

(a) $L \subseteq L^{\mathfrak{t}}$ is $\bar{K}$-closed
(b) $L^* \subseteq L, \bar{A} = \langle A_t : t \in L^* \rangle$
(c) if $t \in L$ and $A \in I^{\mathfrak{t}}_t$ then $A \cap L^* = \emptyset$ (so $(L \backslash L^*) \leq_{\mathfrak{t}} L$)
(d) $A_t \in I^{\mathfrak{t}}_t, A_t \subseteq L$ and A_t is maximal under those requirements
(e) $L \backslash L^*$ is $\bar{K}$-closed (actually follows from clause (d))
(f) A_t is $\bar{K}$-closed

2.3 Claim. *Let $\mathfrak{t}$ be an FSI-template and $\bar{K}$ a $\mathfrak{t}$-memory choice.*
0) The family of $\bar{K}$-closed sets is closed under (arbitrary) unions and intersections. Also if $L \subseteq L^{\mathfrak{t}}$ then $L \cup \bigcup \{K_t : t \in L\}$ is $\bar{K}$-closed.
1) If $L_2 \subseteq L^{\mathfrak{t}}$ is $\bar{K}$-closed and L_1 is an initial segment of L_2, then L_1 is $\bar{K}$-closed.
2) If $L_1 \subseteq L_2 \subseteq L^{\mathfrak{t}}$ are $\bar{K}$-closed then

(α) *$Dp_{\mathfrak{t}}(L_1, \bar{K}) \leq Dp_{\mathfrak{t}}(L_2, \bar{K})$, moreover*
(β) *$(\exists t \in L_2)[L_1 \in I^{\mathfrak{t}}_t]$ implies $Dp_{\mathfrak{t}}(L_1, \bar{K}) < Dp_{\mathfrak{t}}(L_2, \bar{K})$.*

3) If $L_1 \subseteq L_2 \subseteq L^{\mathfrak{t}}$ are $\bar{K}$-closed then $\mathfrak{t} \restriction L_2$ is an FSI-template, L_1 is $(\mathfrak{t} \restriction L_2)$-closed and $Dp_{\mathfrak{t} \restriction L_2}(L_1, \bar{K} \restriction L_2) = Dp_{\mathfrak{t}}(L_1, \bar{K})$. $\square_{2.3}$

Proof. 0), 1) Trivial - read the definitions.
2) We prove by induction on ζ that

$(*)_\zeta(\alpha)$ if $\mathrm{Dp}_{\mathfrak{t}}(L_2, \bar{K}) = \zeta$ (and L_1, L_2 are $\bar{K}$-closed) then $\mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K}) \leq \zeta$
(β) if in addition $(\exists t \in L_2)(L_1 \in I^{\mathfrak{t}}_t)$ then $\mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K}) < \zeta$.

So assume $\mathrm{Dp}_{\mathfrak{t}}(L_2, \bar{K}) = \zeta$, so $\mathrm{Dp}_{\mathfrak{t}}(L_2, \bar{K}) \ngeq \zeta + 1$ hence one of the following cases occurs.

First Case: $\zeta = 0$.
Trivial; note that clause (β) is empty.

Second Case: ζ is odd, L_2 has a $(\mathfrak{t}, \bar{K})$-*representation $(L^*, \bar{A})$ such that $\mathrm{Dp}_{\mathfrak{t}}(L_2 \backslash L^*, \bar{K}) < \zeta$; see Definition 2.2(2).

Let $L_2^- =: L_2 \backslash L^*$; if $L_1 \subseteq L_2^-$ then by the induction hypothesis $\mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K}) \leq \mathrm{Dp}_{\mathfrak{t}}(L_2^-, \bar{K}) < \zeta$, so assume $L_1 \nsubseteq L_2^-$ and so only clause (α) is relevant. Now letting $L_1^- = L_1 \backslash L^*$ we have $[L_1^-, L_2^-$ are $\bar{K}$-closed$]$ & $L_1^- \subseteq L_2^-$ & $\mathrm{Dp}_{\mathfrak{t}}(L_2^-, \bar{K}) < \zeta$ hence $\mathrm{Dp}_{\mathfrak{t}}(L_1^-, \bar{K}) < \zeta$ by the induction hypothesis. Let $L_1^* = L_1 \cap L^*$, so

$L_1^* \subseteq L_1, L_1$ is $\bar K$-closed, $L_1 \backslash L_1^* = (L_2 \backslash L_2^*) \cap L_1$ is $\bar K$-closed, $\mathrm{Dp}_{\mathfrak t}(L_1 \backslash L_1^*, \bar K) = \mathrm{Dp}_{\mathfrak t}(L_1^-, \bar K) < \zeta$. Also easily: $t \in L_1^*$ implies $A_t \cap L_1^-$ is $\bar K$-closed and maximal in $\{A \in I_t^{\mathfrak t} : A \subseteq L_1\}$ so $(L_1^*, \langle A_t \cap L_1 : t \in L_1^* \rangle)$ is a $(\mathfrak t, \bar K)$-*representation of L_1. So clearly $\mathrm{Dp}_{\mathfrak t}(L_1, \bar K) \le \mathrm{Dp}_{\mathfrak t}(L_1^-, \bar K) + 1 \le \zeta$ if $\mathrm{Dp}_{\mathfrak t}(L_1^-, \bar K) + 1$ is odd, and $\mathrm{Dp}_{\mathfrak t}(L_1, \bar K) \le (\mathrm{Dp}_{\mathfrak t}(L_1^-, \bar K) + 1) + 1 \le \zeta$ if $\mathrm{Dp}_{\mathfrak t}(L_1^-, \bar K) + 1$ is even hence $< \zeta$.

<u>Third Case</u>: ζ is even and $\langle L^a : a \in M \rangle$ is a $(\mathfrak t, \bar K)$-representation of L_2 such that $a \in M \Rightarrow \mathrm{Dp}_{\mathfrak t}(L_a, \bar K) < \zeta$.
Let $L_2^a =: L^a$ and $L_1^a =: L^a \cap L_1$, so $\langle L_1^a : a \in M \rangle$ is increasing, $\bigcup\limits_{a \in M} L_1^a = L_1$ and each L_1^a is $\bar K$-closed (as L_2^a, L_1 are $\bar K$-closed, see part (0)) and $t \in L_1 \ \ \& \ \ A \in I_t^{\mathfrak t} \ \ \& \ \ A \subseteq L_1 \Rightarrow (\exists a \in M)(A \subseteq L_2^a \cap L_1 = L_1^a)$. Also by the induction hypothesis, $b \in M \Rightarrow \ \mathrm{Dp}_{\mathfrak t}(L_2^b, \bar K) < \zeta$. By the last two sentences (and Definition 2.1) we get $\mathrm{Dp}_{\mathfrak t}(L_1, \bar K) \le \zeta$, as required in clause (α). For clause (β) we know that there is $t \in L_2$ such that $L_1 \in I_t^{\mathfrak t}$, hence by clause (f) of Definition 2.2(1)) for some $b \in M$ we have $L_1 \subseteq L^b$ and we can use the induction hypothesis on ζ for L_1, L^b.
3) Easy. $\square_{2.3}$

2.4 Claim. *1) If for $\ell = 1,2$ we have $\bar L^\ell$ is a $(\mathfrak t, \bar K)$-representation of L and $\bar L^\ell = \langle L_a^\ell : a \in M_\ell \rangle$ and $M = M_1 \times M_2$ <u>then</u> $\bar L = \langle L_a \cap L_b : (a,b) \in M \rangle$ is a $(\mathfrak t, \bar K)$-representation of L.*
2) If $(L_\ell^*, \bar A^\ell)$ is a $(\mathfrak t, \bar K)-^*$representation of L for $\ell = 1,2$ and we let $L^- = L \backslash L_1^* \backslash L_2^*$ and $\bar A = \langle A_t : t \in L_1^* \cup L_2^* \rangle$ and A_t is A_t^ℓ if $t \in L_\ell^*$ (no contradiction!) <u>then</u>

- (a) $\bar A^1 \restriction (L_1^* \cap L_2^*) = \bar A^2 \restriction (L_1^* \cap L_2^*)$
- (b) $(L_1^* \cup L_2^*, \bar A)$ *is a $(t, \bar K)$-*representation of L.*

Proof. 1) Straight.
2) Easy, too. $\square_{2.4}$

<u>2.5 Discussion</u>: 1) Our next aim is to define iteration for any $\bar K$-smooth FSI-template $\mathfrak t$; for this we define and prove the relevant things; of course, by induction on the depth. In the following Definition 2.6, in clause (A)(a), we avoid relying on [Sh 630]; moreover the reader may consider only the case $K_t = \emptyset$, omit $\underset{\sim}{\eta}_t$ and have $\underset{\sim}{Q}_{t,\bar\varphi'_t}$ be the dominating real forcing = Hechler forcing.
2) We may more generally than here allow $\underset{\sim}{\eta}_t$ to be e.g. a sequence of ordinals, and member of $\underset{\sim}{Q}_{t,\varphi,\underset{\sim}{\eta}_t}$ be $\subseteq \mathscr{H}_{<\aleph_1}(\mathrm{Ord})$, and even K_t large but increasing L, we need more "information" from $\underset{\sim}{\eta}_t \restriction \mathrm{Lim}_{\mathfrak t}(\bar Q \restriction L)$. We may change to: $\underset{\sim}{Q}_t$ is a definition

of nep c.c.c. forcing ([Sh 630]) or just "Souslin c.c.c. forcing (= snep)" or just absolute enough c.c.c. forcing notion. All those cases do not make real problems (but when the parameter $\underset{\sim}{\eta}_t$ have length $\geq \kappa$ it change in the ultrapower! i.e. $\mathbf{j}(\underset{\sim}{\eta}_t)$ has length $>$ length of $\underset{\sim}{\eta}_t$).

3) If we restrict ourselves to σ-centered forcing notions (which is quite reasonable), we may in Definition 2.1(3)(a) omit $\boxtimes_{t,L}$ if in Definition 2.6 below in (A)(b) second case we add that $t \in L^* \Rightarrow p \restriction (L \backslash L^*)$ forces a value to $\underset{\sim}{f}_t(p(t))$ where $\underset{\sim}{f}_t : \underset{\sim}{Q}_t \rightarrow w$ witnessed σ-centerness and is absolute enough (or just assume $Q_t \subseteq \omega \times Q'_t$, $f_t(p(t)$ is the first coordinate). More carefully we can do this with σ-linked instead σ-centered.

2.6 Definition/Claim. Let $\mathfrak{t}$ be an FSI-template and $\bar{K} = \langle K_t : t \in L^{\mathfrak{t}} \rangle$ be a smooth $\mathfrak{t}$-memory choice.

By induction on the ordinal ζ we shall define and prove

(A) [Def] for $L \subseteq L^{\mathfrak{t}}$ which is $\bar{K}$-closed of $(\mathfrak{t}, \bar{K})$-depth $\leq \zeta$ we define

- (a) when $\bar{Q} = \langle \underset{\sim}{Q}_{t,\bar{\varphi}_t,\underset{\sim}{\eta}_t} : t \in L \rangle$ is a $(\mathfrak{t}, \bar{K})$-iteration of def-c.c.c. forcing notions, but we can let $\underset{\sim}{\eta}_t$ code $\bar{\varphi}_t$ so usually omit $\bar{\varphi}_t$
- (b) $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$ for $\bar{Q}$ as in (A)(a)

(B) [Claim] for $L_1 \subseteq L_2 \subseteq L^{\mathfrak{t}}$ which are $\bar{K}$-closed of $(\mathfrak{t}, \bar{K})$-depth $\leq \zeta$ and $(\mathfrak{t}, \bar{K})$-iteration of def-c.c.c. forcing notions $\bar{Q} = \langle \underset{\sim}{Q}_{t,\bar{\varphi}_t}, \underset{\sim}{\eta}_t : t \in L_2 \rangle$ and letting $\bar{Q}^1 = \bar{Q}^2 \restriction L_1$ we prove:

- (a) $\bar{Q} \restriction L_1$ is a $(\mathfrak{t}, \bar{K} \restriction L_1)$-iteration of def-c.c.c. forcing notions
- (b) $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1) \subseteq \mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$ as quasi orders
- (c) if $L_1 \leq_{\mathfrak{t}} L_2$ (see Definition 2.1(6)) and $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q}^2)$, <u>then</u> $p \restriction L_1 \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q}^1)$ and $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q}^2) \models$ "$p \restriction L_1 \leq p$"
- (d) if $L_1 \leq_{\mathfrak{t}} L_2$ and $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$ and $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q}^1 \restriction L_1) \models$ "$(p \restriction L_1) \leq q$" <u>then</u> $q \cup (p \restriction (L_2 \backslash L_1))$ is a lub of $\{p, q\}$ in $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q}^2)$; hence $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$
- (e) $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$, that[2] is
  - (i) $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1) \Rightarrow p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$
  - (ii) $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1) \models p \leq q \Rightarrow \mathrm{Lim}_{\mathfrak{t}}(\bar{Q}) \models p \leq q$
  - (iii) if $\mathscr{I} \subseteq \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1)$ is predense in $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1)$, <u>then</u>

[2] here we do not assume $L_1 \leq_{\mathfrak{t}} L_2$,

$\mathscr{I}$ is predense in $\mathrm{Lim}(\bar{Q})$ (hence if $p, q \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$ are incompatible in $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1)$ then they are incompatible in $\mathrm{Lim}(\bar{Q})$)

(f) if $L_0 \subseteq L_2$ is $\bar{K}$-closed, $L = L_0 \cap L_1$ and $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_0)$ and $q \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L)$ satisfies
$(\forall r \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L))[q \leq r \rightarrow p, r$ are compatible in $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_0)]$
<u>then</u>
$(\forall r \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1))[q \leq r \rightarrow p, r$ are compatible in $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_2)]$

[explanation: this means that if q forces for $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_0)}$ that $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_0)/\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L)$ <u>then</u> q forces for $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1)}$ that $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q})/\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1)$.]

(g) if $\langle L_a : a \in M_1 \rangle$ is a $(\mathfrak{t}, \bar{K})-$representation of L_1 <u>then</u> $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1) = \bigcup\limits_{a \in M_1} \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^1_a)$

(h) if $(L^*, \bar{A})$ is a $(\mathfrak{t}, \bar{K})$-*representation of L_1, <u>then</u> $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1)$ is as defined in (A)(b) of our definition below, second case, from $(L^*, \bar{A})$

(i) (α) if $p_1, p_2 \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$ and $t \in \mathrm{Dom}(p_1) \cap \mathrm{Dom}(p_2) \Rightarrow p_1(t) = p_2(t)$, <u>then</u> $q = p_1 \cup p_2$ (i.e. $p_1 \cup (p_2 \backslash (\mathrm{Dom}(p_1)))$ belongs to $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$ and is a l.u.b. of p_1, p_2

(β) $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$ <u>iff</u> p is a function with domain a finite subset of L_2 such that for every $t \in \mathrm{Dom}(p)$ for some $A \in I^{\mathfrak{t}}_t$, A is $\bar{K}$-closed and $K_t \subseteq A$ and $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction A)}$ "$p(t) \in \underset{\sim}{Q}_{t, \underset{\sim}{\eta}_t}$"

(γ) $\mathrm{Lim}_t(\bar{Q}) \models p \leq q$ <u>iff</u> $p, q \in \mathrm{Lim}_t(\bar{Q})$ and for every $t \in \mathrm{Dom}(p)$ we have $t \in \mathrm{Dom}(q)$ and for some $\bar{K}$-closed $A \in I^{\mathfrak{t}}_t$ we have $q \restriction A \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction A)$ and $q \Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction A)}$ "$p(t) \leq q(t)$ in $Q_{t, \underset{\sim}{\eta}_t}$ (as interpreted in $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction A)}$ of course)"

(j) $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$ is a c.c.c. forcing notion and $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q}) = \cup\{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L) : L \in [L_2]^{\leq \aleph_0}\}$

(k) $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$ has cardinality $\leq |L_2|^{\aleph_0}$ (here we use the assumption that $\underset{\sim}{\eta}_t$ and members of $\underset{\sim}{Q}_{t, \underset{\sim}{\eta}_t}$ are reals; see definition in (A)(a)(i)+(i)) below).

Let us carry the induction.

<u>Part (A)</u>: [Definition]

So assume $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) \leq \zeta$. If $\mathrm{Dp}_{\mathfrak{t}}(L) < \zeta$ we have already defined being $(\mathfrak{t}, \bar{K})$-iteration and $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L)$, so assume $\mathrm{Dp}_{\mathfrak{t}}(L) = \zeta$.

<u>Clause(A)(a)</u>:

(*i*) $\underset{\sim}{\eta}_t$ is a $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction K_t)$-name of a real (i.e. from ${}^\omega 2$, used as a parameter) (legal as $K_t \subseteq L$ & $K_t \in I_t$ & $t \in L$ hence by 2.3(2), clause (β) we have $\mathrm{Dp}_{\mathfrak{t}}(K_t, \bar{K}) <$ $\mathrm{Dp}_{\mathfrak{t}}(K_t \cup \{t\}, \bar{K}) \leq$ $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) \leq \zeta$ so $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_t)$ is a well defined forcing notion by the induction hypothesis and 2.3(2), clause (β))

(*ii*) $\bar{\varphi}_t$ is a pair of formulas with the parameters $\underset{\sim}{\eta}_t$ defining in $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction K_t)}$ a forcing notion denoted by $Q_{t, \bar{\varphi}_t, \underset{\sim}{\eta}_t}$ whose set of elements is $\subseteq \mathscr{H}(\aleph_1)$

(*iii*) in $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction K_t)}$, if $P' \lessdot P''$ are c.c.c. forcing notions <u>then</u> $Q_{t, \bar{\varphi}_t, \eta_t}$ as interpreted in $(\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction K_t)})^{P'}$ is a c.c.c. forcing notion there, and it is a $\lessdot$-subforcing of $(P''/P') * \underset{\sim}{Q}_{t, \bar{\varphi}, \eta_t}$ where $Q_{t, \bar{\varphi}, \eta_t}$ mean as interpreted in $(\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction K_t)})^{P''}$ (i.e. "$p \leq q$","p, q incompatible", "$\langle p_n : n < \omega \rangle$ is predense" (so the sequence is from the smaller universe) are preserved).

<u>Clause (A)(b)</u>:

<u>First Case</u>: $\zeta = 0$.

Trivial

<u>Second Case</u>: $\zeta > 0$ odd.

So let $(L^*, \bar{A})$ be a $(\mathfrak{t}, \bar{K})$-*representation of L.

Define

$$p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q}) \text{ \underline{iff} } p : \begin{array}{l} p \text{ is a finite function, } \mathrm{Dom}(p) \subseteq L, p \restriction (L \backslash L^*) \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction (L \backslash L^*)) \\ \text{and if } t \in L^* \cap \mathrm{Dom}(p), \text{ then } p(t) \text{ is a } \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction A_t)\text{-name} \\ \text{of a member of } Q_{t, \bar{\varphi}_t, \underset{\sim}{\eta}_t} \end{array}$$

and the order is

$\mathrm{Lim}_{\mathfrak{t}}(\bar{Q}) \models p \leq q$ <u>iff</u>

(*i*) $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction (L \backslash L^*)) \models$ "$(p \restriction (L \backslash L^*) \leq (q \restriction (L \backslash L^*))$" and

(*ii*) if $t \in L^* \cup$ $\mathrm{Dom}(p)$ then $q \restriction A_t \Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction A_t)}$ "$p(t) \leq q(t)$".

Clearly $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$ is a quasi order. But we should prove that $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$ is well defined, which means that the definition does not depend on the representation. So we prove

$\boxtimes_1$ if $\mathrm{Dp}_{\mathfrak{t}}(L, \bar{K}) = \zeta$ and for $\ell = 1, 2$ we have $(L^*_\ell, \bar{A}^\ell)$ is a $(\mathfrak{t}, \bar{K})$-*representation of L with $\mathrm{Dp}_{\mathfrak{t}}(L \backslash L^*_\ell, \bar{K}) < \zeta$ and Q^ℓ is $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L)$ as defined by $(L^*_\ell, \bar{A}^\ell)$ above, <u>then</u> $Q^1 = Q^2$.

This is immediate by Claim 2.4(2) and the induction hypothesis clause (B)(h).

Third Case: ζ even > 0.

So there are a directed partial order M and $\bar{L} = \langle L_a : a \in M\rangle$ a $(\mathfrak{t}, \bar{K})$-representation of L such that $a \in M \Rightarrow \mathrm{Dp}_{\mathfrak{t}}(L_a, \bar{K}) < \zeta$. By the induction hypothesis, $a \leq_M b \Rightarrow L_a \subseteq L_b \;\&\; \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_a) \subseteq \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_b)$.

We let $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L) = \bigcup\limits_{a \in M} \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_a)$, so we have to prove

$\boxtimes_2$ the choice is of $\bar{L}$ is immaterial.

So we just assume that for $\ell = 1, 2$ we have: M_ℓ is a directed partial order, $\bar{L}^\ell = \langle L^\ell_a : a \in M_\ell\rangle, L^\ell_a \subseteq L, M_\ell \models a \leq b \Rightarrow L^\ell_a \subseteq L^\ell_b$ and $(\forall t \in L)(\forall A \in I_t)[A \subseteq L \rightarrow (\exists a \in M_\ell)(A \subseteq L^\ell_a)$ and $\mathrm{Dp}_{\mathfrak{t}}(L^\ell_a, \bar{K}) < \zeta$.
We should prove that $\bigcup\limits_{a \in M_1} \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^1_a), \bigcup\limits_{a \in M_2} \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^2_a)$ are equal, as quasi orders of course.

Now $M =: M_1 \times M_2$ with $(a_1, a_2) \leq (b_1, b_2) \Leftrightarrow a_1 \leq_{M_1} b_1 \;\&\; a_2 \leq_{M_2} b_2$, is a directed partial order. We let $L_{(a_1,a_2)} = L^1_{a_1} \cap L^2_{a_2}$, so clearly $L_{(a_1,a_2)} \subseteq L^{\mathfrak{t}}$, $\mathrm{Dp}_{\mathfrak{t}}(L_{(a_1,a_2)}, \bar{K}) < \zeta$ and $(a_1, a_2) \leq_M (b_1, b_2) \Rightarrow L_{(a_1,a_2)} \subseteq L_{(b_1,b_2)}$ and $\langle L_{(a_1,a_2)} : (a_1, a_2) \in M\rangle$ is a $(\mathfrak{t}, \bar{K})$-representation of L by Claim 2.4(1). So by transitivity of equality, it is enough to prove for $\ell = 1, 2$ that $\bigcup\limits_{a \in M_\ell} \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^\ell_a), \bigcup\limits_{(a,b) \in M} \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_{(a,b)})$ are equal as quasi orders. By the symmetry in the situation without loss of generality $\ell = 1$.
Now for every $a \in M_1, \bar{L} = \langle L_{(a,b)} : b \in M_2\rangle$ satisfies: $L^1_a \subseteq L, \mathrm{Dp}(L^1_a) < \zeta, L_{(a,b)} \subseteq L^1_a, L^1_a = \bigcup\limits_{b \in M_2} L_{(a,b)}, b_1 \leq_{M_2} b_2 \Rightarrow L_{(a,b_1)} \subseteq L_{(a,b_2)}$. Also we know that $(\forall t \in L)(\forall A \in I^{\mathfrak{t}}_t)(\exists b \in M_2)(A \subseteq L \rightarrow A \subseteq L_b)$ hence $(\forall t \in L^1_a)(\forall A \in I^{\mathfrak{t}}_t)(A \subseteq L^1_a \rightarrow (\exists b \in M_2)(A \subseteq L_{(a,b)}))$. Hence by the induction hypothesis for clause (B)(g) we have $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^1_a), \bigcup\limits_{b \in L_2} \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_{(a,b)})$ are equal as quasi orders. As this holds for every $a \in M_1$ and M_1 is directed we get $\bigcup\limits_{a \in M_1} \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^1_a), \bigcup\limits_{a \in M_1} \bigcup\limits_{b \in M_2} \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_{(a,b)})$ are equal as quasi orders. But the second is equal to $\bigcup\limits_{(a,b) \in M} \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_{(a,b)})$ so we are done.

Part (B):

First Case: $\zeta = 0$.

Trivial.

Second Case: $\zeta > 0$ is odd.

So by Definition 2.1(3) there are $L^*, \langle A_t : t \in L^* \rangle$ as in the appropriate case there. Let $\langle t^*_i : i < i(*) \rangle$ list L^* with no repetitions. So $\{A \in I^{\mathfrak{t}}_{t^*_i} : A \subseteq L_2\}$ has the maximal member $A_{t^*_i}$ and $t \in L^* \Rightarrow t \notin A_{t^*_i}$ and $\mathrm{Dp}_{\mathfrak{t}}(L^-_2, \bar{K}) \leq \xi =: \zeta - 1$ where $L^-_2 =: L_2 \backslash L^* = L_2 \backslash \{t^*_i : i < i^*\}$, so $i < i(*) \Rightarrow A_{t^*_i} \subseteq L^-_2$, that is $(L^*_2, \bar{A}^1)$ is a $(\mathfrak{t}, \bar{K})$-*representation of L_2 where $L^*_2 = L_2 \backslash L^-_2, \bar{A}^2 = \langle A_{t^*_i} : i < i(*) \rangle$. So we have already defined $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$. We shall use freely the uniqueness in the second case in the definition (A)(b). Let $L^*_1 = L^*_2 \cap L_1, L^-_1 = L_1 \cap L^-_2$ and $\bar{A}^1 = \langle A^1_t : t \in L^*_1 \rangle$, with $A^1_2 = A^2_t \cap L_1$ and $A^2_t = A_t$.

Clause (B)(a):

Easy.

Clause (B)(b):

If $L_1 = L^-_2$ this follows by the definition of $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_2)$.

If $L_1 \subseteq L^-_2$ this is very easy by the induction hypothesis and the previous sentence. Otherwise, clearly $(L^*_1, \bar{A}^1)$ is a $(\mathfrak{t}, \bar{K})$-*representation of L_2 so by clause $(B)(h)$ when $\mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K}) < \zeta$, and by uniqueness proved in part (A) otherwise, we have: $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1)$ is defined as in (A)(b) second case for $(L^* \cap L_1, \bar{A} \restriction L_1)$. By the induction hypthesis, $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^-_1) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^-_2)$ hence for each $i < i(*)$,

- $(*)$ the forcing notion $\underset{\sim}{Q}_{t^*_i, \bar{\varphi}^{t^*_i}, \underset{\sim}{\eta}_{t^*_i}}$ as interpreted in $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^-_1)}$ is a sub-quasi order of the same forcing notion interpreted in $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^-_2)}$.

Looking at the definitions of $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_2)$, $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1)$ using $(L^*_2, \bar{A}^2)$ and $(L^*_1, \bar{A}^1)$, O.K. by the uniqueness we can easily finish.

Clause (B)(c),(d):

Straight.

Clause (e):

If $L_1 \subseteq L^-_2$ then $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^-_2)$ by the induction hypothesis and $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^-_2) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_2)$ by the definition in part (A) so we are done.

So assume that $L_1 \nsubseteq L^-_2$, so $(L^*_1, \bar{A}^2)$ is a $(\mathfrak{t}, \bar{K})$-*representation of L_1 so the definition in clause (A)(b) second case apply. Consider $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^-_1) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^-_2)$ which hold by the induction hypothesis and the definitions of $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^-_\ell)$ according to the $(\mathfrak{t}, \bar{K})-^*$representation $(L^*_\ell, \bar{A}^\ell)$.

Now in $(*)$ above we can add

- $(*)^+$ if $j(1) \leq j(2) \leq i, L_{1,j} = L^-_1 \cup \{t^*_\varepsilon : \varepsilon < j, t^*_\varepsilon \in L_1\}, L_{2,j} = L^-_2 \cup \{t^*_\varepsilon : \varepsilon < j\}, \mathscr{I} \in \mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^-_1)}$ is a predense subset of $\underset{\sim}{Q}_{t^*_i, \bar{\varphi}^{t^*_i}, \underset{\sim}{\eta}_{t^*_i}}$ as interpreted in

$\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q}\restriction L_{1,j(1)})}$, then $\mathscr{I}$ is also a predense subset of $Q_{t^*_i,\bar{\varphi}_{t^*_i},\underset{\sim}{\eta}_{t^*_i}}$ as interpreted in $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q}\restriction L_{2,j(2)})}$ is.

So the conclusion is immediate.

Clause (B)(f):

Let $L_0, L = L_1 \cap L, q \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L), p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_0)$ be as there; clearly $L_0 \cup L_1$ is $\bar{K}$-closed.

If $\mathrm{Dp}_{\mathfrak{t}}(L_0 \cup L_1, \bar{K}) < \zeta$ we can use the induction hypothesis and clause (B)(e) which we have already proved; so assume that this fails, so $\mathrm{Dp}_{\mathfrak{t}}(L \cup L_1, \bar{K}) = \zeta$ and so let $(L^*, \bar{A})$ witness this. Now using $\boxtimes_1$ and the induction hypothesis for clause (B)(h) we can prove it by induction on $i(*)$ thus reducing it to the case $i(*)+1, t^*_i \in L$ which is easy using $(*)^+$ from above so we are done.

Clause (B)(g):

Again using $\boxtimes_1$ and the induction hypothesis for clause (B)(h).

Clause (B)(h):

Straight.

Clause (B)(i):

Easy.

Clause (B)(j):

Let $p_\alpha \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$ for $\alpha < \omega_1$; set $w_\alpha =: \{i : t^*_i \in \mathrm{Dom}(p_\alpha)\}$, so without loss of generality $\langle w_\alpha : \alpha < \omega_1 \rangle$ form a Δ-system with heart w; let $p'_\alpha = p_\alpha \restriction (L^-_2 \cup w)$, and easily it suffices to prove that for some $\alpha \neq \beta, p'_\alpha, p'_\beta$ are compatible in $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction (L^-_2 \cup w))$ (if q is a common upper bound of p'_α, p'_β in $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction (L^-_2 \cup w))$, then $q^+ = q \cup (p'_\alpha \restriction (L_2 \backslash L^-_2 \backslash w)) \cup (p'_\beta \restriction (L_2 \backslash L^-_2 \backslash w))$ is as required by clauses $(\ell), (m)$ which is said below easily holds). We can do this by induction on $|w|$ and (using the uniqueness proved in (A)(b) above) we can reduce this to the case w is a singleton, say $\{t^*_0\}$. So $p^-_\alpha = p'_\alpha \restriction L^-_2 \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^-_2)$ for $\alpha < \omega_1$ hence for some $G_2 \subseteq \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^-_2)$ generic over $\mathbf{V}$, the set $u = \{\alpha < \omega_1 : p^-_\alpha \in G_2\}$ is uncountable; now as $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction A_{t^*_0}) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^-_2)$, clearly $G^* = G_2 \cap \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction A_{t^*_0})$ is generic over $\mathbf{V}$ and $\alpha \in u \Rightarrow p^-_\alpha(t^*_0)[G^*] \in \underset{\sim}{Q}_{t^*_0,\bar{\varphi}_{t^*_0},\eta^*_0}[\mathbf{V}[G^*]] \subseteq \underset{\sim}{Q}_{t^*_0,\bar{\varphi}_{t^*_0},\underset{\sim}{\eta}_{t^*_0}}[Q][\mathbf{V}[G_2]]$.

Hence by (A)(a)(iii) below for some $\alpha \neq \beta$ from $u, p^-_\alpha(t^*_0)[G^*], p^-_\alpha(t^*_0)[G^*]$ are compatible in $\underset{\sim}{Q}_{t^*_0,\bar{\varphi}_{t^*_0},\underset{\sim}{\eta}_{t^*_0}[G^*]}[\mathbf{V}[G_2]]$, hence in $\underset{\sim}{Q}_{t^*_0,\bar{\varphi}_{t^*_0},\underset{\sim}{\eta}_{t^*_0}[G^*]}[\mathbf{V}[G^*]]$, and we can easily finish.

Clause (k),(l),(m):

Easy.

Third Case: $\zeta > 0$ even.

So let $\langle L^2_a : a \in M\rangle$ be a $(\mathfrak{t}, \bar{K})$-representation of L_2 with $a \in M \Rightarrow \mathrm{Dp}_{\mathfrak{t}}(L_a, \bar{K}) < \zeta$.

Clause (B)(a):

Trivial.

Clause (B)(b):

Clearly $\mathrm{Dp}_{\mathfrak{t}}(L_2, \bar{K}) \le \zeta$ by Claim 2.3(2)(α) hence $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1)$ is well defined by (A)(b) above $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q}) = \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_2) = \bigcup\limits_{a \in M_2} \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^2_a)$ as quasi orders.

Clearly $\langle L^1_a = L_1 \cap L^2_a : a \in M\rangle$ is a $(\mathfrak{t}, \bar{K})$-representation of L_1 hence by the induction hypothesis (if $\mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K}) < \zeta$) or by the uniqueness proved in (A)(b) (if $\mathrm{Dp}_{\mathfrak{t}}(L_1, \bar{K}) = \zeta$) we know that $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1) = \bigcup\limits_{a \in M} \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^1_a)$ as quasi orders and by the induction hypothesis for (B)(b) we know $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^1_a) \subseteq \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^2_a)$ as quasi orders (for $a \in M$), and we can easily finish.

Clause (B)(c),(d):

Use the proof of clause (B)(b) noting that $L^1_a \le_{\mathfrak{t}} L^2_a$ and so we can use the induction hypothesis (i.e. if $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_2)$, as M is directed there is $a \in M$ such that $\mathrm{Dom}(p) \subseteq L^2_a$, now $a \le_M b \Rightarrow p \restriction L^1_b = p \restriction L^1_a$ and we can finish easily).

Clause (B)(e):

The statements (i) + (ii) holds by clause (b).
The statement (iii) holds: let $\mathscr{I}$ be a predense subset of $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1)$, let $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$, so for some $a \in M$ we have $p \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^2_a)$. By the induction hypothesis applying clause (B)(e) to p, L^1_a, L^2_a there is $q \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^1_a)$ as there. Now by the assumption on "$\mathscr{I} \subseteq \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1)$ is dense", as $q \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1)$ (by clause (B)(b)) we can find $q_0 \in \mathscr{I}$ and q_1 such that $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_1) \models q_0 \le q_1$ & $q \le q_1$, so for some $b \in M$ we have $q, q_0, q_1 \in L^1_b$ and $a \le_M b$ (as M is directed). Now we consider $p, q, L^1_a, L^2_a, L^1_b, L^2_b$ and apply by clause (B)(f).

Clause (B)(f):

Easy to check using clause (f) for the L^2_a's, which holds by the induction hypothesis.

Clause (B)(g):

Let $M_2 =: M$. For each $a_1 \in M_1$, clearly $\mathrm{Dp}_{\mathfrak{t}}(L_a, \bar{K}) \le \zeta$ as $L_{a_1} \subseteq L_1$ and $\langle L_{a_1} \cap L^2_{a_2} : a_2 \in M_2\rangle$ is a $(\mathfrak{t}, \bar{K})$-representation of L_a hence by (A)(b) we know $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_{a_1}) = \bigcup\limits_{a_2 \in M_2} \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction (L_{a_1} \cap L^2_{a_2}))$. The rest should be clear.

Clause (B)(h):

Easy.

Clause (B)(i):

Easy.

Clause (B)(j):

So let $p_\alpha \in \text{Lim}_{\mathfrak{t}}(\bar{Q})$ for $\alpha < \omega_1$; let $w_\alpha = \text{Dom}(p_\alpha)$ and without loss of generality $\langle w_\alpha : \alpha < \omega_1 \rangle$ is a Δ-system with heart w. So for some $a \in M$ we have $w \subseteq L^2_a$. For each α, for some $a_\alpha \in M$ we have $a \le_M a_\alpha$ and $p_\alpha \in \text{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^2_{a_\alpha})$, so by clause (e) for $L^2_{a_\alpha}, L^2_a$ (which holds by the induction hypothesis, there is $p^+_\alpha \in \text{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^2_a)$ such that $p^*_\alpha \Vdash_{\text{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_a)}$ "$p \in \text{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^2_{a_\alpha})/\text{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^2_a)$" and by the induction hypothesis for some $\alpha < \omega_1$ there is $q \in \text{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^2_a)$ which is (there) above p^*_α and above p^*_β.

Let $q^+ = q \cup (p_\alpha \restriction (w_\alpha \backslash w)) \cup (p_\beta \restriction (w_\beta \backslash w))$ and let $p^+_\alpha = q \cup (p_\alpha \restriction (w_\alpha \backslash w))$ and $p^+_\beta = q \cup (p_\beta \restriction (w_\beta \backslash w))$. Clearly $L_a \le_{\mathfrak{t}} L_2$ hence by clause $(i)(\beta) + (\gamma)$ for $\bar{Q} \restriction (L^2_{a_\alpha} \cup L^2_{a_\beta})$ we have $p^+_\alpha \in \text{Lim}_{\mathfrak{t}}(\bar{Q}), q \le p^+_\alpha, p_\alpha \le p^+_\alpha$ and similarly $p^+_\beta \in \text{Lim}_{\mathfrak{t}}(\bar{Q}), q \le p^+_\beta, p_\beta \le p^+_\beta$ clause $(B)(i)(\alpha)$ our $q = p^+_\alpha \cup p^+_\beta$ is as required.

Clause (k):

Easy. $\square_{2.6}$

2.7 Claim. *1) Assume*

- (a) *$\mathfrak{t}$ is an FSI-template, $Dp_{\mathfrak{t}}(L, \bar{K}) < \infty$ i.e. $\bar{K}$ is a smooth $\mathfrak{t}$-memory choice*
- (b) *$\bar{Q} = \langle \underset{\sim}{Q}_{t,\underset{\sim}{\eta}_t} : t \in L \rangle$ is a $(\mathfrak{t}, \bar{K})$-iteration of def-c.c.c. forcing notions*
- (c) *$L_1, L_2 \subseteq L$ and $L_1 < L_2$ (that is $(\forall t_1 \in L_1)(\forall t_2 \in L_2)(L^{\mathfrak{t}} \models t_1 < t_2)$) and $t \in L_2 \Rightarrow L_1 \in I^{\mathfrak{t}}_t$ or at least $t \in L_2$ & $L' \subseteq L_1$ & $|L'| \le \aleph_0 \Rightarrow L' \in I^{\mathfrak{t}}_t$ and $L = L_1 \cup L_2$.*

Then

- (α) *$Lim_{\mathfrak{t}}(\bar{Q})$ is actually a definition of a forcing (in fact c.c.c. one) so meaningful in bigger universes, moreover for extensions $\mathbf{V}_1 \subseteq \mathbf{V}_2$ of $\mathbf{V} = \mathbf{V}_0$ (with the same ordinals of course), we*[3] *get $[Lim_{\mathfrak{t}}(\bar{Q})]^{\mathbf{V}_1} \lessdot [Lim_{\mathfrak{t}}(\bar{Q})]^{\mathbf{V}_2}$*
- (β) *$Lim_{\mathfrak{t}}(\bar{Q})$ is in fact $Q_1 * \underset{\sim}{Q}_2$ where $Q_1 = Lim_{\mathfrak{t}}(\bar{Q} \restriction L_1)$ and $Q_2 = [Lim_{\mathfrak{t}}(\bar{Q} \restriction L_2)]^{\mathbf{V}[\underset{\sim}{G}_{Q_1}]}$ (composition).*

2) Assume clauses (a), (b) of part (1) and

- $(c)_2$ *L has a last element t^* and let $L^- = L \backslash \{t^*\}$.*

[3] of course possibly $L_1 = \emptyset$

Then for any $G^- \subseteq Lim_{\mathfrak{t}}(\bar{Q} \restriction L^-)$ generic over $\mathbf{V}$, letting $\eta_{t^*} = \underset{\sim}{\eta}_{t^*}[G^-]$ in $\mathbf{V}[G^-]$ we have: the forcing notion $Lim_{\mathfrak{t}}(\bar{Q})/G^-$ is equivalent to $\cup\{Q^{\mathbf{V}[G^-_A]}_{t^*,\eta_{t^*}} : A \in I^{\mathfrak{t}}_{t^*}$ is $\bar{K}$-closed$\}$ where $G^-_A =: G^- \cap Lim_{\mathfrak{t}}(\bar{Q} \restriction A)$ and $\eta_{t^*_1} = \underset{\sim}{\eta}_{t^*}[G^-]$.

3) Assume clauses (a), (b) of part (1) and

- $(c)_3$ *$\langle L_i : i < \delta\rangle$ is an increasing continuous sequence of initial segments of L with union L and δ is a limit ordinal.*

Then $Lim_{\mathfrak{t}}(\bar{Q})$ is $\bigcup\limits_{i<\delta} Lim_{\mathfrak{t}}(\bar{Q} \restriction L_i)$, moreover $\langle Lim_{\mathfrak{t}}(\bar{Q} \restriction L_i) : i < \delta\rangle$ is $\lessdot$-increasing continuous.
4) Assume $\mathfrak{t}^1, \mathfrak{t}^2$ are FSI-templates, $L^{\mathfrak{t}^1} = L^{\mathfrak{t}^2}$ call it L and for every $t \in L, I^{\mathfrak{t}^1}_t \cap [L]^{\le\aleph_0} = I^{\mathfrak{t}^2}_t \cap [L]^{\le\aleph_0}$ and $\bar{K}$ is smooth $\mathfrak{t}^\ell$-memory choice and $\bar{Q} = \langle Q_{t,\bar{\varphi}_t,\underset{\sim}{\eta}_t} : t \in L\rangle$ is a $(\mathfrak{t}^\ell, \bar{K})$-iteration of def-c.c.c. forcing notions for $\ell = 1,2$. Then $Lim_{\mathfrak{t}^1}(\bar{Q}) = Lim_{\mathfrak{t}^2}(\bar{Q})$.

Proof. Straight (or read [Sh 630]). $\square_{2.7}$

We now give sufficient conditions for: "if we force by $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$ from 2.6, then some cardinal invariants are small or equal/bigger than some μ. The necessity of such a claim in our framework is obvious; we deal with two-place relations only as this is the case in the popular cardinal invariants, in particular those we deal with.

2.8 Claim. *Assume $\mathfrak{t}, \bar{K}$ and $\bar{Q} = \langle \underset{\sim}{Q}_{t,\underset{\sim}{\eta}_t} : t \in L^{\mathfrak{t}}\rangle$ are as in 2.6 and $P = Lim_{\mathfrak{t}}(\bar{Q})$.*

1) Assume

- (a) *R is a Borel[4] two-place relation on ${}^\omega\omega$ (we shall use $<^*$)*
- (b) *$L^* \subseteq L^{\mathfrak{t}}$*
- (c) *for every countable $\bar{K}$-closed $A \subseteq L^{\mathfrak{t}}$ for some $t \in L^*$ we have $A \in I^{\mathfrak{t}}_t$*
- (d) *for $t \in L^*$ and $\bar{K}$-closed $A \in K^{\mathfrak{t}}_t$ which include K_t, in $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q}\restriction A)}$ we have $\Vdash_{\underset{\sim}{Q}_{t,\underset{\sim}{\eta}_t}}$ "$\underset{\sim}{\nu}_t \in {}^\omega\omega$ is an R-cover of the old reals, that is $\rho \in ({}^\omega\omega)^{\mathbf{V}} \Rightarrow \rho R \underset{\sim}{\nu}_t$" where $\underset{\sim}{\nu}_t$ is a name in the forcing $\underset{\sim}{Q}_{t,\underset{\sim}{\eta}_t}$ i.e. in $(Q_{t,\eta_t[\underset{\sim}{G}]})^{\mathbf{V}[\underset{\sim}{G}]}$, $\underset{\sim}{G}$ the generic subset of $Lim_{\mathfrak{t}}(\bar{Q} \restriction A)$; not depending on A. (Usually $\underset{\sim}{\nu}_t$ is the generic real of $\underset{\sim}{Q}_{t,\underset{\sim}{\eta}_t}$, and hence $\underset{\sim}{Q}_{t,\underset{\sim}{\eta}_t}$ is interpreted in the universe $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q}\restriction A)}$, so $\underset{\sim}{\eta}_t$ is determined by the generic; normally this we assume absolutely).*

[4] here and below just enough absoluteness is enough, of course

Then $\Vdash_P$ "$(\forall \rho \in {}^\omega\omega)(\exists t \in L^*)(\rho R \underset{\sim}{\nu}_t)$, *i.e.* $\{\underset{\sim}{\nu}_t : t \in L^*\}$ *is an* R*-cover, which, if* $R = <^*$ *means* $\mathfrak{d} \leq |L^*|$".
1A) If we weaken assumption (d) to "for some $\underset{\sim}{\nu}_t$ *a* $Lim_{\mathfrak{t}}(\bar{Q} \restriction K_t)$*-name" we get* $\Vdash_P$ "$(\forall \rho \in {}^\omega\omega)(\exists t)(\exists \nu \in \mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction K_t)})[\rho R \nu]$.
2) Assume

- (a) R *is a Borel two-place relation on* ${}^\omega\omega$ *(we shall use* $<^*$*)*
- (b) μ *is a cardinality*
- (c) *if* $L^* \subseteq L^{\mathfrak{t}}, |L^*| < \mu$ *then for some* $t \in L^{\mathfrak{t}}$ *and* $\bar{K}$*-closed* $L^{**} \supseteq L^*$ *we have* $L^{**} \in I^{\mathfrak{t}}_t$ *and in* $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^{**})}, \Vdash_{Q_{t,\underset{\sim}{\eta}_\delta}}$ "$\underset{\sim}{\nu}_t$ *is a* R*-cover of the old reals" with* $\underset{\sim}{\nu}_t$ *some* $Q_{t,\underset{\sim}{\nu}_t}$*-name as in (1); (usually* $\underset{\sim}{\nu}_t$ *is the generic real of* $Q_{t,\underset{\sim}{\nu}_1}$ *(this we assume absolutely).*

Then $\Vdash_P$ "$(\forall X \in [{}^\omega\omega]^{<\mu})(\exists \nu \in {}^\omega\omega)(\bigwedge\limits_{\rho \in X} \rho R \nu)$"
(so for $R = <^*$ *this means* $\mathfrak{b} \geq \mu$*).*
3) Assume

- (a) R *is a Borel two-place relation*[5] *on* ${}^\omega\omega$ *(we use* $R = \{(\rho, \nu) : \rho, \nu \in {}^\omega 2$ *and* $\rho^{-1}\{1\}, \nu^{-1}\{1\}$ *are infinite with finite intersection)*
- (b) κ *a cardinality,* $cf(\kappa) > 2^{\aleph_0}$ *and* $\kappa < \lambda$
- (c) *if* $t_{i,n} \in L^{\mathfrak{t}}$ *for* $i < i(*), n < \omega$ *and* $\kappa \leq i(*) < \lambda$ *and each:* $\{t_{i,n} : n < \omega\}$ *is* $\bar{K}$*-closed, then we can find* $t_n \in L^{\mathfrak{t}}$ *for* $n < \omega$ *such that* $\{t_n : n < \omega\} \subseteq L^{\mathfrak{t}}$ *is* $\bar{K}$*-closed and:*
  - $(*)$ *for every* $i < i(*)$ *for some* $j < \kappa, j \neq i$ *and the mapping* $t_{i,n} \mapsto t_{i,n}, t_{j,n} \mapsto t_n$ *is a partial isomorphism of* $(\mathfrak{t}, \bar{Q})$ *(see 2.9 below).*

Then in $\mathbf{V}^P$ *we have*

- $\boxtimes^R_\mu$ *if* $\rho_i, \nu_i \in {}^\omega\omega$ *for* $i < i(*)$ *and* $\mu \leq i(*) < \lambda$ *and* $i \neq j \Rightarrow \nu_i R \rho_j$, *then we can find* $\rho \in {}^\omega\omega$ *such that* $i < i(*) \Rightarrow \nu_i R \rho$.

Proof. Straight, but being requested:
1) Let $\underset{\sim}{\rho}$ be a P-name of a member of $({}^\omega\omega)^{\mathbf{V}^P}$, so as P satisfies (see 2.4(B)(j)), for

[5] so R is defined in $\mathbf{V}$; if R is from $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction K)}$, we need partial isomorphism (see below) of $(\mathfrak{t}, \bar{Q})$ extending id_K

each n there is a maximal antichain $\{p_{n,i} : i < i_n\}$ such that $p_{n,i}$ forces a value of $\underset{\sim}{\rho}(n)$ and, of course, i_n is countable. Let $M = \{a : a$ is a countable $\bar{K}$-closed subset of $L^{\mathfrak{t}}\}$, so obviously M is closed under countable unions and $\cup\{a : a \in M\} = L^{\mathfrak{t}}$; and let $L_a = a$ for $a \in M$ so by 2.4(B)(g) we have $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q}) = \cup\{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_a) : a \in M\}$ but $P = \mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$, hence for $n < \omega, i < i_n$ for some $a_{n,i} \in M$ we have $p_{n,i} \in \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_a)$. But M is $\aleph_1$-directed so for some $a \in M$ we have $\{a_{n,i} : n < \omega, i < i_n\} \subseteq \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_a)$. Also by 2.4(B)(e) we know $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_a) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{Q}) = P$, so $\underset{\sim}{\rho}$ is a $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L_a)$-name. Now by assumption (c) of what we are proving, as $L_a \subseteq L$ is countable, we can find $t \in L^* \subseteq L^{\mathfrak{t}}$ such that $L_a \in I^{\mathfrak{t}}_t$. Also we know that $K_t \in I^{\mathfrak{t}}_t$ (see Definition 2.1(2)(c) hence $A =: K_t \cup L_a$ belongs to $I^{\mathfrak{t}}_t$ and is $\bar{K}$-closed; and easily also $B = A \cup \{t\}$ is $\bar{K}$-closed.

So $A \subseteq B \subseteq L^{\mathfrak{t}}$ are $\bar{K}$-closed so as above $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction A) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction B) \lessdot \mathrm{Lim}_{\mathfrak{t}}(\bar{Q}) = P$ and $\underset{\sim}{\rho}$ is a $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction A)$-name (hence also a $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction B)$) of a member of ${}^\omega\omega$.

Now by assumption (d) in $\mathbf{V}^{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction A)}$ we have $\Vdash_{Q_{t,n_i}}$ "$\underset{\sim}{\rho} R \underset{\sim}{\nu}_t$", hence by 2.4(B)(h) we know that $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction B) = \mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction A) * \underset{\sim}{Q}_{t,\underset{\sim}{n}_t}$, so together $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(B)}$ "$\underset{\sim}{\rho} R \underset{\sim}{\nu}_t$" hence the previous sentence and obvious absoluteness we have $\Vdash_P$ "$\underset{\sim}{\rho} R \underset{\sim}{\nu}_t$". So as $\underset{\sim}{\rho}$ was any P-name of a member of $({}^\omega\omega)^{\mathbf{V}^P}$ we are done.

1A) Same proof.

2) So assume $p \Vdash_P$ "$\underset{\sim}{X} \subseteq {}^\omega\omega$ has cardinality $< \mu$". As we can increase p without loss of generality for some $\theta < \mu$ we have $p \Vdash_P$ "$|\underset{\sim}{X}| = \theta$" so we can find a sequence $\langle \underset{\sim}{\rho}_\alpha : \alpha < \theta\rangle$ of P-names of members of $({}^\omega\omega)^{\mathbf{V}^P}$ such that $p \Vdash_P$ "$\underset{\sim}{X} = \{\underset{\sim}{\rho}_\alpha : \alpha < \theta\}$". Let $\{p_{\alpha,n,i} : i < i_{\alpha,n}\}$ be a maximal antichain of P, with $p_{\alpha,n,i}$ forcing a value to $\underset{\sim}{\rho}_\alpha(n)$ and $i_{\alpha,n}$ countable.

Define $M = \{a \subseteq L^{\mathfrak{t}}$: a countable $\bar{K}$-closed$\}$, so for each $\alpha < \theta, n < \omega, i < i_{\alpha,n}$ for some $a_{\alpha,n,i} \in M$ we have $p_{\alpha,n,i} \in \mathrm{Lim}_t(\bar{Q} \restriction L_a)$. So for some $\bar{K}$-closed $L^{**} \subseteq L^{\mathfrak{t}}$ and $t \in L^{\mathfrak{t}}$ we have $L^{**} \in I^{\mathfrak{t}}_t$ and $a_{\alpha,n,i} \subseteq L^{**}$ for $\alpha < \theta, n < \omega, i < i_{\alpha,n}$. We now continue as in part (1).

3) So assume $i(*) \in [\kappa, \lambda)$ and $\Vdash_P$ "$\underset{\sim}{\nu}_i, \underset{\sim}{\rho}_i \in {}^\omega\omega$ and $i \neq j \Rightarrow \underset{\sim}{\nu}_i R \underset{\sim}{\rho}_j$". So as above we can find countable $\bar{K}$-closed $K^*_i \subseteq L^{\mathfrak{t}}$ such that $\underset{\sim}{\nu}_i, \underset{\sim}{\rho}_i$ are $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction K^*_i)$-names; without loss of generality $K^*_i \neq \emptyset$ and even $|K^*_i| = \aleph_0$; this is impossible only if $L^{\mathfrak{t}}$ is finite and then all is trivial. Let $\langle t_{i,n} : n < \omega\rangle$ be a list of the members of K^*_i with no repetitions. Let $f_{i,j}$ be the mapping from K^*_j to K^*_i defined by $f_{i,j}(t_{j,n}) = t_{i,n}$.

We define a two place relation E_1, E_2 on $i(*)$ and on $i(*) \times i(*)$ respectively

$$iE_1j \text{ \underline{iff} } f_{i,j} \text{ is a partial isomorphism of } (\mathfrak{t},\bar{Q})$$
$$\text{such that } \hat{f}_{i,j} \text{ maps } (\underset{\sim}{\rho}_j,\underset{\sim}{\nu}_j) \text{ to } (\underset{\sim}{\rho}_i,\underset{\sim}{\nu}_i)$$

$$(i_1,i_2)E_2(j_1,j_2) \text{ \underline{iff} } i_1E_1j_1, i_2E_2j_2 \text{ and}$$
$$f_{i_1,j_1} \cup f_{i_2,j_2} \text{ is a partial isomorphism of } (\mathfrak{t},\bar{Q}).$$

Easily

$\boxtimes(i)$ E_1,E_2 are equivalence classes over their domain

(ii) E_1,E_2 has $\le 2^{\aleph_0}$ equivalence classes

(iii) $f_{j,i} = f^{-1}_{i,j}$.

As $|i(*)/E_1| \le 2^{\aleph_0} <$ cf(κ) (by $(*)(ii)$ and assumption (b) respectively) and we can replace $i(*)$ by $i(*)+\kappa$, without loss of generality $i<\kappa \Rightarrow 0E_1i$. Now we apply assumption (c), and get $\langle t_n : n<\omega\rangle$. By $(*)$ of clause (c) for any i,j clearly $K^*_i \cup K^*_j$ and $K^*_i \cup \{t_n : n<\omega\}$ are $\bar{K}$-closed (see the definition below). For any $i<i(*)$ let $j_i<\kappa$ be as in $(*)$ of clause (c) which means: $j_i \ne i$ and the following mapping g_i is a partial isomorphism of $(\mathfrak{t},\bar{Q}) =$ Dom$(g_i) = \{t_{i,n},t_{j_i,n} : n<\omega\}, g_i(t_{i,n}) = t_{i,n}, g_i(t_{j,n}) = t_n$.

Let $\underset{\sim}{\nu},\underset{\sim}{\rho}$ be $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction K^*)$-names such that for some equivalently any i, $\hat{g}_i$ maps $\underset{\sim}{\nu}_{j_i},\underset{\sim}{\rho}_{j_i}$ to $\underset{\sim}{\nu},\underset{\sim}{\rho}$ respectively (this is O.K. as for any i_1,i_2 we have $j_{i_1}E_1j_{i_2}$ because j_{i_1},j_{i_2} hence $g_{i_2} \circ f_{j_2,j_{i_1}} = g_{i_1} \restriction K^*_{j_{i_1}}$). Now for any $i<\mu$, as $j_i \ne i$, we know $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(\bar{Q}\restriction(K^*_i \cup K^*_{j_i}))}$ "$\underset{\sim}{\nu}_i R \underset{\sim}{\rho}_{j_i}$", so applying g_i we have $\Vdash_{\mathrm{Lim}_{\mathfrak{t}}(K^*_i \cup K^*)}$ "$\underset{\sim}{\nu}_i R \underset{\sim}{\rho}$". So we have proved $\boxtimes^R_\mu$. $\square_{2.8}$

In 2.9 we note that isomorphisms (or embeddings) of $\mathfrak{t}$'s tend to induce isomorphisms (or embeddings) of $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$, and deal (in 2.10,2.11) with some natural operation. In 2.9 we could use two $\mathfrak{t}$'s, but this can trivially be reduced to one.

2.9 Definition/Claim. Assume that $\mathfrak{t},\bar{K}$ and $\bar{Q} = \langle \underset{\sim}{Q}_{t,\eta_t} : t \in L^{\mathfrak{t}}\rangle$ are as in 2.6. By induction on ζ we define and prove[6]

[6] if $K_t = \emptyset$ and all $\underset{\sim}{Q}_{t,\eta}$ have the same definition of forcing notion, as in our main case, we can separate the definition and claim

(A) [Def] we say f is a partial isomorphism of $(\mathfrak{t}, \bar{Q})$ if: (writing $\mathfrak{t}$ instead of $(\mathfrak{t}, \bar{Q})$ means we assume $Q_{t,\underset{\sim}{\eta}_t} = Q$, i.e. constant)

- (a) f is a partial one-to-one function from $L^{\mathfrak{t}}$ to $L^{\mathfrak{t}}$
- (b) Dom(f), Rang(f) are $(\mathfrak{t}, \bar{K})$-closed sets of depth $\leq \zeta$
- (c) for $t \in$ Dom(f) and $A \subseteq$ Dom(f) we have $A \in I^{\mathfrak{t}}_t \Leftrightarrow f''(A) \in I^{\mathfrak{t}}_{f(t)}$
- (d) for $t \in$ Dom(f), we have: f maps K_t onto $K_{f(t)}$ and $f \restriction K_t$ maps $\underset{\sim}{\eta}_t$ to $\underset{\sim}{\eta}_{f(t)}$, more exactly the isomorphism which f induces from $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction K_t)$ onto $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction K_{f(t)})$ does this.

(B) [Claim] f induces naturally an isomorphism which we call $\hat{f}$ from $\mathrm{Lim}(\bar{Q} \restriction$ Dom$(f))$ onto $\mathrm{Lim}_{\mathfrak{t}}(\bar{Q} \restriction$ Rang$(f))$.

Proof. Straightforward.

2.10 Definition. 1) We say $\mathfrak{t} = \mathfrak{t}^1 + \mathfrak{t}^2$ if

- (a) $L^{\mathfrak{t}} = L^{\mathfrak{t}^1} + L^{\mathfrak{t}^2}$ (as linear orders)
- (b) for $t \in L^{\mathfrak{t}^1}, I^{\mathfrak{t}^1}_t = I^{\mathfrak{t}}_t$
- (c) for $t \in L^{\mathfrak{t}^2}, I^{\mathfrak{t}^2}_t = \{A \subseteq L^{\mathfrak{t}} : A \cap L^{\mathfrak{t}^2} \in I^{\mathfrak{t}^2}_t\}$.

So $\mathfrak{t}^1 + \mathfrak{t}^2$ is well defined if $\mathfrak{t}^1, \mathfrak{t}^2$ are disjoint, i.e. $L^{\mathfrak{t}^1} \cap L^{\mathfrak{t}^2} = \emptyset$.
2) We say $\mathfrak{t}^1 \leq_{wk} \mathfrak{t}^2$ iff

- (a) $L^{\mathfrak{t}^1} \subseteq L^{\mathfrak{t}^2}$ (as linear orders)
- (b) for $A \subseteq L^{\mathfrak{t}^1}$ and $t \in L^{\mathfrak{t}^1}$ we have $A \in I^{\mathfrak{t}^1}_t \Leftrightarrow A \in I^{\mathfrak{t}^2}_t$.

3) If $\langle \mathfrak{t}^\zeta : \zeta < \xi \rangle$ is $\leq_{wk}$-increasing, ξ a limit ordinal, we define $\mathfrak{t}^\xi =: \bigcup\limits_{\zeta<\xi} \mathfrak{t}^\zeta$ by

$$L^{\mathfrak{t}^\xi} = \bigcup_{\zeta<\xi} L^{\mathfrak{t}^\zeta} \qquad \text{(as linear orders)}$$

$$I^{\mathfrak{t}^\xi}_t = \cup\{I^{\mathfrak{t}^\zeta}_t : \zeta < \xi \text{ and } t \in L^{\mathfrak{t}_\zeta}\}$$

Clearly $\zeta < \xi \Rightarrow \mathfrak{t}^\zeta \leq_{wk} \mathfrak{t}^\xi$. Such $\mathfrak{t}^\xi$ is called the limit of $\langle \mathfrak{t}^\zeta : \zeta < \xi \rangle$; now a $\leq_{wk}$-increasing sequence $\langle \mathfrak{t}^\zeta : \zeta < \xi \rangle$ is called continuous if for every limit ordinal

$\delta < \xi$ we have $\mathfrak{t}^\delta = \bigcup\limits_{\zeta<\delta} \mathfrak{t}^\zeta$.

4) If $\langle \mathfrak{t}^\zeta : \zeta < \xi \rangle$ are pairwise disjoint (that is $\zeta \neq \varepsilon \Rightarrow L^{\mathfrak{t}^\zeta} \cap L^{\mathfrak{t}^\varepsilon} = \emptyset$) we define $\sum\limits_{\zeta<\xi} \mathfrak{t}^\zeta$ by induction on ξ naturally: for $\xi = 1$ it is $\mathfrak{t}^0$, for ξ limit it is $\bigcup\limits_{\zeta<\xi} (\sum\limits_{\zeta<\varepsilon} \mathfrak{t}^\zeta)$ and for $\xi = \varepsilon + 1$ it is $(\sum\limits_{\zeta<\varepsilon} \mathfrak{t}^\zeta) + \mathfrak{t}^\varepsilon$, so $\xi_1 \leq \xi_2 \Rightarrow \sum\limits_{\zeta<\xi_1} \mathfrak{t}^\zeta \leq_{wk} \sum\limits_{\zeta<\xi_2} \mathfrak{t}^\zeta$ (even an initial segment).
5) We can replace in 0) - 4) above $\mathfrak{t}^\zeta$ by $(\mathfrak{t}^\zeta, \bar{K}^\zeta)$.

2.11 Claim. *Let $\mathfrak{t}$ be an FSI-template.*
1) If $L^\mathfrak{t} = \emptyset$ or just $L^\mathfrak{t}$ is finite <u>then</u> $\mathfrak{t}$ is smooth.
2) If $\mathfrak{t}^1, \mathfrak{t}^2$ are disjoint FSI-templates, <u>then</u> $\mathfrak{t}^1 + \mathfrak{t}^2$ is a FSI-template and $\rho \in \{1,2\} \Rightarrow \mathfrak{t}^\ell \leq_{wk} \mathfrak{t}^1 + \mathfrak{t}^2$.
3) If $\mathfrak{t}^1, \mathfrak{t}^2$ are disjoint smooth FSI-templates <u>then</u> $\mathfrak{t}^1 + \mathfrak{t}^2$ is a smooth FSI-template.
4) If $\langle \mathfrak{t}^\zeta : \zeta < \xi \rangle$ is an $\leq_{wk}$-increasing (2.10(2)) sequence of FSI-templates and ξ is a limit ordinal, <u>then</u> $\mathfrak{t}^\xi =: \bigcup\limits_{\zeta<\xi} \mathfrak{t}^\zeta$ is an FSI-template and $\zeta < \xi \Rightarrow \mathfrak{t}^\zeta \leq_{wk} \mathfrak{t}^\xi$.
5) If $\langle \mathfrak{t}^\zeta : \zeta < \xi \rangle$ is an increasing continuous (see Definition 2.10(3)) sequence of smooth FSI-templates and ξ is a limit ordinal, <u>then</u> $\mathfrak{t}^\xi =: \bigcup\limits_{\zeta<\xi} \mathfrak{t}^\zeta$ is a smooth FSI-template and $\zeta < \xi \Rightarrow \mathfrak{t}^\zeta \leq_{wk} \mathfrak{t}^\xi$.
6) If $\langle \mathfrak{t}^\zeta : \zeta < \xi \rangle$ is a sequence of pairwise disjoint [smooth] FSI-templates, <u>then</u> $\sum\limits_{\zeta<\xi} t^\zeta$ is a [smooth] FSI-template and $\langle \sum\limits_{\zeta<\varepsilon} t^\zeta : \varepsilon \leq \zeta \rangle$ is increasing continuous.
7) We can add $\bar{K}^\zeta$ to $\mathfrak{t}^\zeta$.
8) We can restrict ourselves to locally countable $\mathfrak{t}$'s (so the sums are locally countable if the summands are).

Proof. Easy.

* * *

<u>2.12 Discussion</u>: To prove our desired result CON($\mathfrak{a} > \mathfrak{d}$) we need to construct an FSI-template $\mathfrak{t}$ of the right form. Now we do it using a measurable cardinal. The point is that if we are given $\langle \langle t_{i,n} : n < \omega \rangle : i < i(*) \rangle, L^\mathfrak{t}, i(*) \geq \kappa$ and D is a normal ultrafilter on κ, then in $\mathfrak{t}^\kappa/D, \langle \langle t_{i,n} : i < \kappa \rangle/D : n < \omega \rangle$ is as required in 2.8(3)(c), considering $\mathfrak{t}^\kappa/D$ an extension of $\mathfrak{t}$.

2.13 Definition. For a template $\mathfrak{t}$ and an $(2^{\aleph_0})^+$-complete ultrafilter D on κ we define $\mathfrak{t}^* =: \mathfrak{t}^\kappa/D$ as follows:

$$L^{\mathfrak{t}^*} = (L^{\mathfrak{t}})^\kappa/D \text{ as a linear order}$$

and if $t^* = \langle t_i : i < \kappa\rangle/D$ where $t_i \in L^{\mathfrak{t}}$ then we let $I^{\mathfrak{t}^*}_{t^*} = \{A : \text{we can find } A_i \in I^{\mathfrak{t}}_{t_i}$ for $i < \kappa$ such that $A \subseteq \prod\limits_{i<\kappa} A_i/D\}$. We then let $\mathbf{j}_{D,\mathfrak{t}}$ be the canonical embedding of $\mathfrak{t}$ into $\mathfrak{t}^\kappa/D$ and $\mathfrak{t}' = \mathbf{j}_{D,\mathfrak{t}}(\mathfrak{t})$ is defined by $L^{\mathfrak{t}'} = L^{t^*} \restriction \{\mathbf{j}_{d,\mathfrak{t}}(s) : s \in L^{\mathfrak{t}}\}, I^{\mathfrak{t}'}_s = I^{\mathfrak{t}^*}_s \restriction L^{\mathfrak{t}'}$. [We can deal with $\bar K$, if D if $(\bigcup\limits_{t \in L} |K_t|^+)$-complete and can deal also with $\bar Q$ if we have $<$ com(D) kinds of $\bar\varphi_t$.]

2.14 Claim. *1) In 2.13, $\mathfrak{t}^\kappa/D$ is also a template and $\mathbf{j}_{D,\mathfrak{t}}(\mathfrak{t}) \le_{wk} \mathfrak{t}^\kappa/D$.*
2) If $\mathfrak{t}$ is a smooth FSI-template <u>then</u> $\mathfrak{t}^\kappa/D$ is a smooth FSI-template.

Proof. Straight.

Now 2.15, 2.16 below are used only in the short proof of 2.17 depending on §1, so you may ignore them.

2.15 Definition. Fix $\aleph_0 < \kappa < \mu = \text{ cf}(\mu) < \lambda = \text{ cf}(\lambda) = \lambda^\kappa$ and D a κ-complete (or just $(2^{\aleph_0})^+$-complete) uniform ultrafilter on κ. We define by induction on $\zeta \le \lambda$, smooth FSI-template $\mathfrak{t}_{\gamma,\zeta}$ for $\gamma < \mu$ such that:

- (a) $\mathfrak{t}_{\gamma,\zeta}$ is a locally countable FSI-template
- (b) if $\gamma_1 \ne \gamma_2$ then $\mathfrak{t}_{\gamma_1,\zeta}, \mathfrak{t}_{\gamma_2,\zeta}$ are disjoint, i.e. $L^{\mathfrak{t}_{\gamma_1,\zeta}} \cap L^{\mathfrak{t}_{\gamma_2,\zeta}} = \emptyset$
- (c) for $\xi < \zeta$ we have $\mathfrak{t}_{\gamma,\xi} \le_{wk} \mathfrak{t}_{\gamma,\zeta}$
- (d) if ζ is limit then $\mathfrak{t}_{\gamma,\zeta} = \bigcup\limits_{\xi<\zeta} \mathfrak{t}_{\gamma,\xi}$, see 2.10(3), 2.11(6).
- (e) if $\zeta = \xi + 1$ and ξ is even, <u>then</u> there is an isomorphism $\mathbf{j}_{\gamma,\zeta}$ from $\sum\limits_{\beta \le \gamma} \mathfrak{t}_{\beta,\xi}$ onto $\mathfrak{t}_{\gamma,\zeta}$ which is the identity over $\mathfrak{t}_{\gamma,\xi}$
- (f) if $\zeta = \xi + 1$ and ξ is odd, <u>then</u> there is an isomorphism $\mathbf{j}_{\gamma,\zeta}$ from $(\mathfrak{t}_{\gamma,\xi})^\kappa/D$ onto $\mathfrak{t}_{\gamma,\zeta}$ which extends the inverse of $\mathbf{j}_{D,\mathfrak{t}_{\gamma,\xi}}$.

<u>2.16 Observation</u>: The definition is 2.15 is legitimate.

Proof. By the previous claims.

<u>2.17 Conclusion</u>: Assume: κ is measurable, $\kappa < \mu = \mathrm{cf}(\mu) < \lambda = \mathrm{cf}(\lambda) = \lambda^\kappa$. <u>Then</u> for some c.c.c. forcing notion P of cardinality λ, in $\mathbf{V}^P$ we have $\mathfrak{a} = \lambda, \mathfrak{b} = \mathfrak{d} = \mu$.

Proof. <u>Short Proof</u> (depending on §1). Let $\mathfrak{t}_{\gamma,\zeta}$ (for $\gamma < \mu, \zeta \leq \lambda$) be as in 2.15. Let $\mathfrak{t} = \sum\limits_{\gamma<\mu} \mathfrak{t}_{\gamma,\lambda}$ and let $\bar{K} = \langle K_t : t \in L^{\mathfrak{t}}\rangle, K_t = \emptyset$ and let $\bar{Q} = \langle \underset{\sim}{Q}_t : t \in L^{\mathfrak{t}}\rangle$ with $\underset{\sim}{Q}_t$ being constantly the dominating real forcing (= Hechler forcing). Lastly let $P = \mathrm{Lim}_{\mathfrak{t}}(\bar{Q})$.
The rest is as in the end of §1.

<u>Alternative presentation, self contained not depending on 2.15, 2.16</u>: We define an FSI-template $\mathfrak{t}^\zeta$ for $\zeta \leq \lambda$ by induction on ζ.

<u>Case 1</u>: For $\zeta = 0$.
Let $\mathfrak{t}^\zeta$ be defined as follows:

$$L^{\mathfrak{t}^\zeta} = \mu$$

$$I_\alpha^{\mathfrak{t}^\zeta} = \{A : A \subseteq \alpha \text{ is countable}\}$$

<u>Case 2</u>: For $\zeta = \xi + 1$.
We choose $\mathfrak{t}^\zeta$ such that there is an isomorphism $\mathbf{j}_\zeta$ from $L^{\mathfrak{t}^\zeta}$ onto $(L^{\mathfrak{t}^\xi})^\kappa/D$, such that $\mathbf{j}_\zeta \restriction L^{\mathfrak{t}^\xi}$ is the canonical embedding $\mathbf{j}_{D,\mathfrak{t}^\xi}$, and if $x \in L^{\mathfrak{t}^\zeta}, \mathbf{j}_\zeta(x) = \langle x_\varepsilon : \varepsilon < \kappa\rangle/D \in (L^{\mathfrak{t}^\xi})^\kappa/D$ then:

$$\begin{aligned} A \in I_x^{\mathfrak{t}^\zeta} \ \underline{\text{iff}} \text{ for some } \bar{A} = \langle A_\varepsilon : \varepsilon < \kappa\rangle \text{ we have} & \\ A_\varepsilon \in I_{x_\varepsilon}^{\mathfrak{t}^\xi} \text{ and} & \\ \{\mathbf{j}_\zeta(y) : y \in A\} \subseteq \{\langle y_\varepsilon : \varepsilon < \kappa\rangle/D : \{\varepsilon < \kappa : y_\varepsilon \in A_\varepsilon\} \in D\} & \end{aligned}$$

<u>Case 3</u>: ζ limit.
We choose $\mathfrak{t}^\zeta$ as follows:

$$L^{\mathfrak{t}^\zeta} = \bigcup_{\xi<\zeta} L^{\mathfrak{t}^\xi} \text{ as linear orders}$$

$$I_x^{\mathfrak{t}^\zeta} \text{ is } \{A : A \subseteq \{s : L^{\mathfrak{t}^\zeta} \models \text{“}s < x\text{”}\}\}$$

if $x \in L^{\mathfrak{t}^0}$ and is otherwise[7]

$$\{A : \text{for some } \xi < \zeta \text{ we have } x \in L^{\mathfrak{t}^\xi} \text{ and if } y = \operatorname{Min}\{y \in L^{\mathfrak{t}^0} : L^{\mathfrak{t}^\zeta} \models x < y\}$$
$$\text{which is } \in L^{\mathfrak{t}^0} \text{ then}$$
$$A \backslash \{x \in L^{\mathfrak{t}^\zeta} : L^{\mathfrak{t}^\zeta} \models x < z \text{ for some } z \text{ such that } L^{\mathfrak{t}^0} \models z < y\} \text{ belongs to } I_x^{\mathfrak{t}^\xi}\}.$$

We now prove by induction on $\zeta \le \lambda$ that:

$(*)(a)$ $\mathfrak{t}^\zeta$ is an FSI-template

(b) $L^{\mathfrak{t}^0}$ is an unbounded subset of $L^{\mathfrak{t}^\zeta}$

(c) $\mathfrak{t}^\zeta$ is smooth

(d) $\mathfrak{t}^\xi \le_{wk} \mathfrak{t}^\zeta$ for $\xi < \zeta$

(e) if $y \in L^{\mathfrak{t}^\zeta}$ then $\{z :$ for some $x \in L^{\mathfrak{t}^0}$ we have $L^{\mathfrak{t}^\zeta} \models z \le x$ and $L^{\mathfrak{t}^\zeta} \models x < y\} \in I_x^{\mathfrak{t}^\zeta}$

(f) $L^{\mathfrak{t}^\zeta}$ has cardinality $\le (\mu + |\zeta|)^\kappa$.

Lastly let for $\zeta \le \lambda, P_\zeta = \operatorname{Lim}_{\mathfrak{t}}(\bar{Q} \restriction L^{\mathfrak{t}^\zeta})$. Now

(α) P_λ is a c.c.c. forcing notion of cardinality $\le \lambda^{\aleph_0}$ hence $\mathbf{V}^{P_\lambda} \models 2^{\aleph_0} \le \lambda$ by 2.4(B)(j)

(β) in $\mathbf{V}^{P_\lambda}$ we have $\mathfrak{d} \le \mu$, by 2.8(1) applied with $R =<^*$ and $L^* = L^{\mathfrak{t}^0}$ using $(*)$(b)+(e)

(γ) in $\mathbf{V}^{P_\lambda}$ we have $\mathfrak{b} \ge \mu$ by 2.8(2) applied with $R =<^*$

(δ) $\mathfrak{b} = \mathfrak{d} = \mu$ and $\mathfrak{a} \ge \mu$ by $(\beta) + (\gamma)$ as it is well known that $\mathfrak{b} \le \mathfrak{d}$ and $\mathfrak{b} \le \mathfrak{a}$.

But why the demand (c) from 2.8(3) holds? So assume $i(*) \in [\kappa, \lambda)$ and $t_{i,n} \in L^{\mathfrak{t}^\lambda}$ for $i < i(*), n < \omega$ be given. As λ is regular $> i(*)$, necessarily for some $\xi < \lambda$ we have $\{t_{i,n} : i < i(*), n < \omega\} \subseteq L^{\mathfrak{t}^\xi}$. Now let $t_n \in L^{t^{\xi+1}}$ be such that $\mathbf{j}_{\xi+1}(t_n) = \langle t_{i,n} : i < \kappa \rangle / D$; so

(ε) in $\mathbf{V}^{P_\lambda}$ we have $\mathfrak{a} \ge \kappa \Rightarrow \mathfrak{a} \ge \lambda$ by 2.8(3), see there.

Together we are done. $\square_{2.17}$

[7]this is the "veteranity privilege", i.e. "founding father right"; members t of $L^{\mathfrak{t}^0}$ have the maximal $I_t^{\mathfrak{t}^\zeta}$.

§3 ELIMINATING THE MEASURABLE

Without a measurable cardinal our problem is to verify condition (c) in 2.8(3). Toward this it is helpful to show that for some $\aleph_1$-complete filter D on κ, for any $i(*) \in [\kappa, \lambda)$ and $t_{i,n} \in L^{\mathfrak{t}}$, for $i < i(*), n < \omega$, we have: for every $j < i(*)$ for some $A \in D^+$ we have for any $i_0, i_1 \in A$, the mapping $t_{j,n} \mapsto t_{j,n}$; $t_{i_0,n} \mapsto t_{i_1,n}$ is a partial isomorphism of $\mathfrak{t}$. So D behaves as an $\aleph_1$-complete ultrafilter for our purpose.
[If you know enough model theory, this is the problem of finding convergent sequences, see [Sh 300, Ch.II]; for stable first order T with $\kappa = \kappa_r(T)$ any indiscernible sequence (equivalently set) $\langle \bar{a}_\alpha : \alpha < \alpha^* \rangle$ of cardinality $\geq \kappa$, is convergent; why? as for any $\bar{\mathbf{b}} \in {}^{\kappa >}\mathfrak{C}$, for all but $< \kappa$ ordinals $\alpha < \alpha^*$, $\bar{\mathbf{b}}\hat{\ }\bar{a}_\alpha$ has a fixed type so average is definable. In [Sh 300, Ch.II], we deal with it in general, (so harder to prove existence which we do there under the relevant assumptions).]

3.1 Lemma. *Assume $2^{\aleph_0} < \mu =$ cf$(\mu) < \lambda =$ cf$(\lambda) = \lambda^{\aleph_0}$. Then for some P we have*

- (a) *P is a c.c.c. forcing notion of cardinality λ*
- (b) *in $\mathbf{V}^P$ we have $\mathfrak{b} = \mathfrak{d} = \mu$ and $\mathfrak{a} = 2^{\aleph_0} = \lambda$.*

Proof. We rely on 2.6 + 2.8. Let L_0^+ be a linear order isomorphic to λ, let L_0^- be a linear order anti-isomorphic to λ (and $L_0^- \cap L_0^+ = \emptyset$) and let $L_0 = L_0^- + L_0^+$.

Let $\mathbf{J}$ be the following linear order:

- (a) its set of elements is ${}^{\omega >}(L_0)$
- (b) the order is: $\eta <_{\mathbf{J}} \nu$ iff for some $n < \omega$ we have $\eta \restriction n = \nu \restriction n$ and $\ell g(\eta) = n \ \ \& \ \ \nu(n) \in L_0^+$ or $\ell g(\nu) = n \ \ \& \ \ \eta(n) \in L_0^-$ or we have $\ell g(\eta) > n \ \& \ \ell g(\nu) > n \ \& \ L_0 \models \eta(n) < \nu(n)$.

[See more on such orders [Lv] and [Sh 220, AP], but we are self contained.] ? Lv ?
Note that

- $\boxtimes$ every interval of $\mathbf{J}$ has cardinality λ
- $\boxtimes^+$ if $\aleph_0 < \theta =$ cf$(\theta) < \lambda$ or $\theta = 1$ and $\langle t_i : i < \theta \rangle$ is a strictly decreasing sequence in $\mathbf{J}$ then $\mathbf{J} \restriction \{y \in \mathbf{J} : (\forall i < \theta)(y <_{\mathbf{J}} t_i)\}$ has cofinality λ
- $\boxtimes^-$ the inverse of $\mathbf{J}$ satisfies $\boxtimes^+$, moreover is isomorphic to $\mathbf{J}$.

We now define by induction on $\zeta < \lambda$ an FSI-templates $\mathfrak{t}_\zeta$ such that

- $(*)^1_\zeta$ the set of members of $\mathfrak{t}_\zeta$ is a set of finite sequences starting with ζ hence disjoint to $\mathfrak{t}_\varepsilon$ for $\varepsilon < \zeta$; for $x \in \mathfrak{t}_\zeta$ let $\xi(x) = \zeta$.

<u>Defining $\mathfrak{t}_\zeta$</u>: <u>Case 1</u>: $\zeta = 0, \zeta$ successor or $\mathrm{cf}(\zeta) = \aleph_0$.
Let $L^{\mathfrak{t}_\zeta} = \{\langle\zeta\rangle\}$ and $I^{\mathfrak{t}_\zeta}_{<\zeta>} = \{\emptyset\}$.

<u>Case 2</u>: $\mathrm{cf}(\zeta) > \aleph_0$.
Let $h_\zeta : \mathbf{J} \to \zeta$ be a function such that: $\varepsilon < \zeta \Rightarrow h^{-1}_\zeta\{\varepsilon\}$ is a dense subset of $\mathbf{J}$. The set of elements of $\mathfrak{t}_\zeta$ is

$$\{\langle\zeta\rangle\} \cup \{\langle\zeta\rangle\hat{\ }\langle\eta\rangle\hat{\ }x : \eta \in \mathbf{J} \text{ and } x \in \bigcup_{\varepsilon \le h_\zeta(\eta)} L^{\mathfrak{t}_\varepsilon}\}$$

order $<_{\mathfrak{t}_\zeta}$ defined by:

$$\langle\zeta\rangle \text{ is maximal}$$

$$\begin{aligned}\langle\zeta\rangle\hat{\ }\langle\eta_1\rangle\hat{\ }x_1 <_{\mathfrak{t}_\zeta} \langle\zeta\rangle\hat{\ }\langle\eta_2\rangle\hat{\ }x_2 \text{ \underline{iff} } \eta_1 <_{\mathbf{J}} \eta_2 \vee (\eta_1 = \eta_2 \ \ &\ \ \xi(x_1) < \xi(x_2)) \vee (\eta_1 = \eta_2 \\ &\ \ \xi(x_1) = \xi(x_2) \ \ \&\ \ x_1 <_{\mathfrak{t}_{\xi(x_1)}} x_2).\end{aligned}$$

Lastly, for $y \in \mathfrak{t}_\zeta$ we define the ideal $I = I^{\mathfrak{t}_\zeta}_y$:

(α) if $y = \langle\zeta\rangle$ then
$I = \{Y : Y \subseteq L^{\mathfrak{t}_\zeta} \backslash \{\langle\zeta\rangle\}\}$

(β) if $y = \langle\zeta\rangle\hat{\ }\langle\nu\rangle\hat{\ }x$, <u>then</u> I is the family of sets Y satisfying the following conditions:

(i) $Y \subseteq L^{\mathfrak{t}_\zeta}$

(ii) $(\forall z \in Y)(z <_{\mathfrak{t}_\zeta} y)$

(iii) we each $\eta \in \mathbf{J}$ and $\xi < \zeta$ we have:

$$\{z : \langle\zeta\rangle\hat{\ }\langle\eta\rangle\hat{\ }z \in Y \text{ and } \xi(z) = \xi \text{ and } z \neq \langle\xi\rangle\} \in I^{\mathfrak{t}_\xi}_{\langle\xi\rangle}$$

(iv) the set $\{\eta \in \mathbf{J} : (\exists x)(\langle\zeta\rangle\hat{\ }\langle\eta\rangle\hat{\ }x \in Y)\}$ is finite.

Why is $\mathfrak{t}_\zeta$ really a FSI-template? We prove, of course, by induction on ζ that:

$(*)^2_\zeta$ (i) $L^{\mathfrak{t}_\zeta}$ is a linear order

(ii) $I^{\mathfrak{t}_\zeta}_t$ is an ideal of subsets of $\{s \in I^{\mathfrak{t}_\zeta}_t : s < t\}$

(iii) $\mathfrak{t}_\zeta$ is an FSI-template,

(iv) $\mathfrak{t}_\zeta$ is disjoint to $\mathfrak{t}_\varepsilon$ for $\varepsilon < \zeta$.

[Why? By 2.11.]
Next we prove by induction on ζ, that $\mathfrak{t}_\zeta$ is a smooth FSI-template. Arriving to ζ

$(*)^3_\zeta$ for $\eta \in \mathbf{J}$ and $\varepsilon \le h_\zeta(\eta)+1$, we have $\mathfrak{t}_\zeta \restriction \{\langle\zeta\rangle{}^\frown\langle\eta\rangle{}^\frown\rho : \rho \in \bigcup\limits_{\xi<\varepsilon} \mathfrak{t}_\xi\}$ is a smooth FSI-template.
[Why? We prove by induction on ε; for $\varepsilon = 0$ by 2.11(1), for ε successor by 2.11(3) for ε limit by 2.11(5) and 2.11(6)]

$(*)^4_\zeta$ for $Z \subseteq \mathbf{J}$ we have $\mathfrak{t}_\zeta \restriction (\bigcup\limits_{\eta\in Z}\{\langle\zeta,\eta\rangle{}^\frown\rho : \rho \in \bigcup\limits_{\xi<h_\zeta(\eta)} t_\xi\})$ is a smooth FSI-template.
[Why? By induction on $|Z|$, for $|Z|=0, |Z|=n+1$ by 2.11(3), for $|Z| \ge \aleph_0$ by 2.11(5)]

$(*)^5_\zeta$ $\mathfrak{t}_\zeta \restriction (L^{\mathfrak{t}_\zeta} \backslash \{\langle\zeta\rangle\})$ is a smooth FSI-template.
[Why? By $(*)^4_\zeta$ for $Z = \mathbf{J}$.]

$(*)^6_\zeta$ $\mathfrak{t}_\zeta$ is a smooth FSI-template
[Why? by 2.11(3)]

$(*)^7_\zeta$ if $K \subseteq L^{\mathfrak{t}_\zeta}$ and $t \in L^{\mathfrak{t}_\zeta}$ then the ideal $I^{\mathfrak{t}_\zeta}_t \cap \mathscr{P}(K)$ is generated by a countable family of subsets of κ
[Why? Check by induction on ζ.]

Now for $\zeta \le \lambda$ let

$\boxtimes$ $\mathfrak{s}_\zeta =: \sum\limits_{\varepsilon<\zeta} \mathfrak{t}_\varepsilon$, i.e.

  (i) the set of elements of $\mathfrak{s}_\zeta$ is $\bigcup\limits_{\varepsilon<\zeta} L^{\mathfrak{t}_\varepsilon}$

  (ii) for $x, y \in \mathfrak{s}_\zeta$ we have $x <_{\mathfrak{s}_\zeta} y$ iff $\xi(x) < \xi(y) \vee (\xi(x) = \xi(y)\ \&\ x <_{\mathfrak{t}_\zeta} y)$

  (iii) $I^{\mathfrak{s}_\zeta}_y = \{Y \subseteq \mathfrak{s}_\zeta : (\forall z \in Y)(z <_{\mathfrak{s}_\zeta} y)$ and $\{z \in \mathfrak{s}_\zeta : \xi(z) = \xi(y)$ and $z \in Y\} \in I^{\mathfrak{t}_{\xi(z)}}_y\}$

$(*)^8_\zeta$ $\mathfrak{s}_\zeta$ is a smooth FSI-template.
[Why? Just easier than the proof above.]

$(*)^9_\zeta$ if $K \subseteq L^{\mathfrak{s}_\zeta}$ is countable and $t \in L^{\mathfrak{s}_\zeta}$, <u>then</u> the ideal $I^{\mathfrak{s}_\zeta}_t \cap \mathscr{P}(K)$ of subsets of K is generated by a countable family of subsets of K
[Why? By $(*)^7_\zeta$ and the definition of $\mathfrak{s}_\zeta$.]

Let[8] $\theta = (2^{\aleph_0})^+$, we shall prove below by induction on ζ that $\mathfrak{s}_\zeta, \mathfrak{t}_\zeta$ are (λ, θ)-good (see definition below and Subclaim 3.4) then we can finish the proof as in 2.17 using $\mathfrak{s}_\mu$ (and $(*)^7_\zeta + (*)^9_\zeta$) where

3.2 Definition. 1) We say $\mathfrak{t}$ is (λ, θ, τ)-good if:

$\oplus$ assume that $t_{\alpha,n} \in L^{\mathfrak{t}}$ for $\alpha < \theta, n < \omega, \{t_{\alpha,n} : n < \omega\}$ is $\bar{K}$-closed and $\mathscr{W}$ a family of subsets of ω such that $2^{|\mathscr{W}|} < \theta$, then we can find a club C of θ and a pressing down function h on C such that:

$\oplus'$ if $S \subseteq C$ is stationary in $\theta, (\forall \delta \in S)[\text{cf}(\delta) > \aleph_0]$ and $h \restriction S$ is constant then:

$\boxtimes^1_S$ for every $\alpha < \beta$ in S the truth value of the following statements does not depend on (α, β):
(but may depend on n, m and $w \in \mathscr{W}$)

(i) $t_{\alpha,n} = t_{\beta,m}$

(ii) $t_{\alpha,n} <_{L^{\mathfrak{t}}} t_{\beta,m}$

(iii) $\{t_{\alpha,\ell} : \ell \in w\} \in I^{\mathfrak{t}}_{t_{\alpha,m}}$

(iv) $\{t_{\beta,\ell} : \ell \in w\} \in I^{\mathfrak{t}}_{t_{\alpha,n}}$

(v) $\{t_{\alpha,\ell} : \ell \in w\} \in I^{\mathfrak{t}}_{t_{\beta,n}}$

$\boxtimes^2_S$ let $\delta^* \leq \theta$, $\text{cf}(\delta^*) = \tau, sup(S \cap \delta^*) = \delta^*$; if $\theta \leq \beta^* < \lambda$ and $s_{\beta,n} \in L^{\mathfrak{t}}$ for $\beta < \beta^* < \lambda, n < \omega$ then we can find $t_n \in L^{\mathfrak{t}}$ for $n < \omega$ such that $\{t_n : n < \omega\}$ is $\bar{K}$-closed and for every $\beta < \beta^*$, for every large enough $\alpha \in S \cap \delta^*$ for some $\mathfrak{t}$-partial isomorphism f we have $f(t_n) = t_{\alpha,n}, f(s_{\beta,n}) = s_{\beta,n}$.

2) We say $\mathfrak{t}$ is strongly (λ, θ, τ)-good if above we allow $\mathscr{W} = \mathscr{P}(\omega)$ (so if $\theta > \beth_2$ this is the same). In both cases we may omit τ if $\tau = \theta$.

3.3 Observation: Instead "h regressive" it is enough to demand: for some sequence $\langle X_\alpha : \alpha < \theta \rangle$ of sets, increasing continuous, $|X_\alpha| < \theta$ and for every (or club of) $\delta < \theta$, if $\text{cf}(\delta) > \aleph_0$ then $h(\delta) \in \mathscr{H}_{<\aleph_0}(X_\delta)$.

[8] but if you like to avoid using $(*)^7_\zeta, (*)^9_\zeta$ and $\mathscr{W}$ below just use $\theta = \beth_2^+$. In fact even without $(*)^7_\zeta + (*)^9_\zeta$ above, countable $\mathscr{W}$ suffice but then we have to weaken the notion of isomorphisms, and no point

3.4 Subclaim. *In the proof of 3.1;*

(i) $\mathfrak{t}_\zeta$ *is* (λ, θ)*-good*

(ii) $\mathfrak{s}_\zeta$ *is strongly* $(\lambda, \theta, \aleph_1)$*-good*

(iii) *if cf*$(\zeta) \neq \theta$ *then also* $\mathfrak{s}_\zeta$ *is strongly* (λ, θ)*-good.*

Proof. Recall that $\theta = (2^{\aleph_0})^+$, and let $\mathscr{W}$ be given ($2^{|\mathscr{W}|} < \theta$ for the first version; $\mathscr{W} = \mathscr{P}(\omega)$ for the second, using $(*)^7_\zeta + (*)^9_\zeta$ from the proof of 3.1). We prove this by induction on ζ.

<u>For $\mathfrak{s}_\zeta$</u>:

If $\zeta = 0$ it is empty. Otherwise given $t_{\alpha,n} \in \mathfrak{s}_\zeta = \sum\limits_{\varepsilon<\zeta} \mathfrak{t}_\varepsilon$ for $\alpha < \theta, n < \omega$ let $h^*_0(\delta)$ be the sequence consisting of: $\xi_{\alpha,n} =:$ Min$\{\xi : \xi \in \{\xi(t_{\beta,m}) : \beta < \delta, m < \omega\} \cup \{\infty\}$ and $\xi \geq \xi(t_{\alpha,n})\}$ for $n < \omega$ and $u_\alpha = \{(n,m) : \xi(t_{\alpha,n}) = \xi_{\alpha,m})\}$ and $\mathbf{w}_\alpha = \{(n,w) : n < \omega, w \in \mathscr{W}$ and $\{t_{\alpha,m} : m \in w\} \in I^{\mathfrak{t}}_{t_{\alpha,n}}\}$; that is $h^*_0(\delta) = \langle u_\alpha, \langle \xi_{\alpha,n} : n < \omega\rangle, \mathbf{w}_\alpha\rangle$. If $S_y = \{\delta : \text{cf}(\delta) \geq \aleph_1, h^*_0(\delta) = y\}$ is stationary we define $h^*_1 \restriction S_y$ such that it codes $h^*_0(\delta)$ and if $n(*) < \omega, \alpha \in S_y \Rightarrow \xi(t_{\alpha,n(*)}) = \xi_{\alpha,n(*)}$ call it $\xi_{y,n(*)}$ let $u_{y,n(*)} = \{n : \xi_{\alpha,n} = \xi_{y,n(*)}\}$, <u>then</u> $h_1 \restriction S_y$ codes a function witnessing the (λ,θ)-goodness of $\mathfrak{t}_{\xi_{y,n(*)}}$ for $\langle t_{\alpha,n} : n \in u_{y,n(*)}, \alpha \in S_y\rangle$.

It is easy to check that this shows $\boxtimes^1_S$ even if cf$(\zeta) = \theta$. But assume cf$(\zeta) \neq \theta$ & $\delta^* = \theta$ or $\delta^* < \theta$, cf$(\delta^*) = \aleph_1$ (or just $\aleph_0 <$ cf$(\delta^*) < \theta$), $\delta^* = \sup(S \cap \delta^*)$; we shall prove also the statement from $\boxtimes^2_S$. Let $w_1 = \{n : \langle \xi(t_{\beta,n}) : \beta \in S\rangle$ is strictly increasing$\}$, $w_0 = \{n : \langle \xi(t_{\beta,n}) : \beta \in S\rangle$ is constant$\}$, let $\xi(S,n) = \xi_{S,n} = \cup\{\xi(t_{\beta,n}) : \beta \in S\}$ as cf$(\zeta) \neq \theta$ it is $< \zeta$.

Given $\bar{s} = \langle s_{\beta,n} : n < \omega\rangle$ we have to find $\langle t_n : n < \omega\rangle$ as required in $\boxtimes^2_S$. If $n \in w_0, w'_{0,n} = \alpha\{m \in w_0 : \xi(t_{\alpha,n}) = \xi(t_{\alpha,m})$ for $\alpha \in S\}$ and to choose $\langle t_m : m \in w'_{0,n}\rangle$ we use the induction hypothesis on $\mathfrak{t}_{\xi(S,n)}$. If $n \in w_1$ then we can find $t^*_n \in \mathfrak{t}_{\xi_{S,n}}$ such that $\{t : t \in \mathfrak{t}_{\xi_{S,n}}, t \leq_{\mathfrak{t}_{\xi(S,n)}} t^*\}$ is disjoint to $\{t_{\beta,m} : \beta < \delta^*, m < \omega\} \cup \{s_{\beta,m} : \beta < \beta^*$ and $m < \omega\}$ because the lower cofinality of $L^{\mathfrak{t}_{\xi(S,n)}}$ is the same as that of L_0 and is $\lambda > \theta + |\beta^*|$. We choose $\eta^* \in \mathbf{J}$ such that $(\forall x)(\langle \zeta, \eta^*\rangle \hat{} \langle x\rangle \in \mathfrak{t}_{\xi(S,n)} \Rightarrow t <_{\mathfrak{t}_{\xi(S,n)}} t^*)$ and we choose together $\langle t_{n'} : n' \in w_1, \xi_{S,n'} = \xi_{S,n}\rangle$ taking care of $\mathscr{W}$, (inside and automatically for others, i.e. considering t_{n_1}, t_{n_2} such that $\xi_{S,n_1} \neq \xi_{S,n_2}$), this is immediate.

$\square_{3.1}$

<u>For $\mathfrak{t}_\zeta$</u>:

Similar. $\square_{3.4}$

* * *

We may like to have "$2^{\aleph_0} = \lambda$ is singular", $\mathfrak{a} = \lambda, \mathfrak{b} = \mathfrak{d} = \mu$. Toward this we would like to have a linear order $\mathbf{J}$ such that if $\bar{x} = \langle x_\alpha : \alpha < \kappa \rangle$ is monotonic, say decreasing then for any $\sigma < \lambda$ for some limit $\delta < \kappa$ of uncountable cofinality the linear order $\{y \in \mathbf{J} : \alpha < \delta \Rightarrow y <_{\mathbf{J}} x_\alpha\}$ has cofinality $> \sigma$. Moreover, δ can be chosen to suit ω such sequences $\bar{x}$ simultaneously. So every set of ω-tuples from $\mathbf{J}$ of cardinality $\geq \kappa$ but $< \lambda$ can be "inflated".

3.5 Lemma. *Assume*

(a) $(2^{\aleph_0})^+ < \mu = cf(\mu) \leq \tau < \lambda = \lambda^{\aleph_0}, \lambda$ *singular*

(b) $(\forall \alpha < \tau)[|\alpha|^{\aleph_0} < \mu = cf(\tau)]$

(c) $\tau \geq \aleph_{\mathrm{cf}(\lambda)}$ *or at least*

$(c)^-$ *there is* $f : \lambda \to cf(\lambda)$ *such that if* $\langle \alpha_\varepsilon : \varepsilon < \tau \rangle$ *is strictly increasing continuous,* $\alpha_\varepsilon < \lambda$ *and* $\gamma < cf(\lambda)$ *then for some* $\varepsilon < \tau$ *we have* $f(\alpha_\varepsilon) \geq \gamma$.

Then for some c.c.c. forcing notion of cardinality λ *we have* $\Vdash_P$ "$2^{\aleph_0} = \lambda, \mathfrak{b} = \mathfrak{d} = \kappa, \mathfrak{a} = \lambda$".

Proof. Note that $(c) \Rightarrow (c)^-$, just let $\alpha < \lambda \ \& \ \mathrm{cf}(\alpha) = \aleph_\varepsilon \ \& \ \varepsilon < \mathrm{cf}(\lambda) \Rightarrow f(\alpha) = \varepsilon$, clearly there is such a function and it satisfies clause $(c)^-$. So we can assume $(c)^-$. Let $\theta = (2^{\aleph_0})^+$ and $\sigma = \mathrm{cf}(\lambda)$ and $\langle \lambda_\varepsilon : \varepsilon < \sigma \rangle$ be a strictly increasing sequence of regular cardinals $> \tau$ with limit λ. Let $\langle L_{0,\gamma} : \gamma < \mathrm{cf}(\lambda) \rangle$ be increasing with $\gamma, L_{0,\gamma}$ like L_0 in the proof of 3.1 with λ_ε instead of λ, such that $\beta < \gamma \Rightarrow L_{0,\beta}$ is an interval of $L_{0,\gamma}$. Let $L_0 = \bigcup\limits_{\gamma < \mathrm{cf}(\lambda)} L_{0,\gamma}$ define $g : L_0 \to \mathrm{cf}(\lambda)$ by $g(x) = \mathrm{Min}\{\gamma : x \in L_{0,\gamma}\}$ and let

$$\mathbf{J}^* = \left\{\eta \in {}^{\omega>}(L_0) : \eta(0) \in L_{0,0} \text{ and } \eta(n+1) \in L_{0,g(\eta(n))} \text{ for } n < \omega\right\}$$

ordered as in the proof of 3.5.

We define $\mathfrak{s}_\zeta, \mathfrak{t}_\zeta$ as there. We then prove that $\mathfrak{s}_\zeta, \mathfrak{t}_\zeta$ are (τ, θ)-good and (λ, τ)-good as there and this suffices repeating the proof of 3.1. $\square_{3.7}$

<u>3.6 Discussion</u>: We may like to separate $\mathfrak{b}$ and $\mathfrak{d}$. So below we adapt the proof of 3.1 to do this (can do it also for 3.5).

A way to do this is to look at the forcing in 3.1 as the limit of the FS iteration $\langle P^*_i, \underset{\sim}{Q}^*_j : i \leq \mu, j < \mu \rangle$, so the memory of Q^*_j is $\{i : i < j\}$ where $\underset{\sim}{Q}^*_j$ is $\mathrm{Lim}_{\mathfrak{t}}[\langle Q_t :$

$t \in L^{\mathfrak{t}_j}\rangle]$. Below we will use the limit of FS iteration $\langle P^*_i, \underset{\sim}{Q}^*_j : j < \mu \times \mu_1\rangle, Q^*_\zeta$ has memory $w_\zeta \subseteq \zeta$ where e.g. for $\zeta = \mu\,\alpha + i, w_\zeta = \{\kappa\beta + j : \beta \le \alpha, j \le i, (\beta, j) \ne (\alpha, i)\}$. Let $P^* = P^*_{\mu \times \mu_i}$ be $\cup\{P_i : i < \mu \times \mu_1\}$.

Of course, Q_ζ will be defined as $\mathrm{Lim}_{\mathfrak{t}_\zeta}(\bar{Q})$, the $\mathfrak{t}_\zeta$ defined as above and $\mathfrak{b} = \mu, \mathfrak{d} = \mu_1$. Should be easy. If $\langle \underset{\sim}{A}_\varepsilon : \varepsilon < \varepsilon^{\bar{x}}\rangle$ exemplifies $\mathfrak{a}$ in $\mathbf{V}^{P^*}$, so $\varepsilon^* \ge \mu$ then for some $(\alpha^*, \beta^*) \in \mu \times \mu_1$ for $\kappa(= \theta)$ of the names they involve $\{\underset{\sim}{Q}_{\mu\alpha+\beta} : \alpha \le \alpha^*, \beta \le \beta^*\}$ only.

Using indiscernibility on the pairs (α, β) to making them increase we can finish.

3.7 Lemma. *1) In Lemma 3.1, if $\mu = cf(\mu) \le cf(\mu_1), \mu_1 < \lambda$, then we can change in the conclusion $\mathfrak{b} = \mathfrak{d} = \mu$ to $\mathfrak{b} = \mu, \mathfrak{d} = \mu_1$.*
2) Similarly for 3.5.

Proof. First Proof: If μ_1 regular, let $\mu_0 = \mu$. The proof of 3.1 for $\ell \in \{0,1\}$ using $\mu = \mu_\ell$ gives $\mathfrak{s}^\ell_{\mu_\ell}$ and without loss of generality $\mathfrak{s}^0_{\mu_0}, \mathfrak{s}^1_{\mu_1}$ are disjoint. Let $\mathfrak{s}$ be $\mathfrak{s}_0 +' \mathfrak{s}_1$ meaning $L[\mathfrak{s}] = L[\mathfrak{s}^0_{\mu_0}] + L[\mathfrak{s}^1_{\mu_1}]$, and for $t \in L[\mathfrak{s}^\ell_{\mu_\ell}]$ we let $I^{\mathfrak{s}}_t =: I^{\mathfrak{s}^\ell_{\mu_\ell}}_t$ (this is not $\mathfrak{s}_0 + \mathfrak{s}_1$ of 2.11. Now the appropriate goodness can be proved.

Second Proof: Instead of starting with $\langle Q_i : i < \mu\rangle$ with full memory we start with $\langle \underset{\sim}{Q}_\zeta : \zeta < \mu \times \mu_1\rangle, \underset{\sim}{Q}_\zeta$ with memory if $\zeta = \mu\alpha + i, i < \kappa, w_\zeta = \{\mu\beta + j : \beta \le \alpha, j \le i, (\beta, j) \ne (\alpha, i)\}$. $\square_{3.7}$

§4 On related cardinal invariants

4.1 Theorem. *Assume*

(a) κ *is a measurable cardinal*

(b) $\kappa < \mu = cf(\mu) < \lambda = cf(\lambda) = \lambda^\kappa$.

Then for some c.c.c. forcing notion P of cardinality λ, in $\mathbf{V}^P$ we have: $2^{\aleph_0} = \lambda, \mathfrak{u} = \mathfrak{d} = \mathfrak{b} = \mu$ and $\mathfrak{a} = \lambda$.

Remark. Recall $\mathfrak{u} = \text{Min}\{|\mathscr{P}| : \mathscr{P} \subseteq [\omega]^{\aleph_0}$ generates a nonprincipal ultrafilter on $\omega\}$.

Proof. The proof is broken to definitions and claims.

4.2 Definition. For an ultrafilter D on ω let $Q(D)$ be:
$\{T : T \subseteq {}^{\omega>}\omega$ is closed under initial segments, and for some $\text{tr}(T)$, the trunk of T, we have:

(i) $\ell \leq \ell g(tr(T)) \Rightarrow T \cap {}^\ell\omega = \{tr(T) \restriction \ell\}$

(ii) $\text{tr}(T) \trianglelefteq \eta \in {}^{\omega>}\omega \Rightarrow \{n : \eta\char`\^\langle n\rangle \in T\} \in \underset{\sim}{D}_i\}$

ordered by inverse inclusion.

4.3 Definition. Let $\mathfrak{K}$ be the family of $\mathfrak{t}$ consisting of $\bar{Q} = \bar{Q}^{\mathfrak{t}} = \langle P_i, \underset{\sim}{Q}_i : i < \mu\rangle = \langle P^{\mathfrak{t}}_i, \underset{\sim}{Q}^{\mathfrak{t}}_i : i < \mu\rangle$ and $\bar{D} = \bar{D}^{\mathfrak{t}} = \langle \underset{\sim}{D}_i : i < \mu$ and $\text{cf}(i) \neq \kappa\rangle$ and $\bar{\tau}^{\mathfrak{t}} = \langle \underset{\sim}{\tau}^{\mathfrak{t}}_i : i < \mu\rangle$ such that

(a) $\bar{Q}$ is a FS-iteration of c.c.c. forcing notions (and $P_\mu = P^{\mathfrak{t}}_\mu = \text{Lim}(\bar{Q}^{\mathfrak{t}}) = \bigcup\limits_{i<\mu} P^{\mathfrak{t}}_i$)

(b) if $i < \mu$, $\text{cf}(i) \neq \kappa$ then $Q_i = Q(\underset{\sim}{D}_i)$,
see Definition 4.2 above

(c) $\underset{\sim}{D}_i$ is a P_i-name of a nonprincipal ultrafilter on ω when $i < \mu$, $\text{cf}(i) \neq \kappa$

(d) $|P_i| \leq \lambda$

(e) for $i < \mu$, $\text{cf}(i) \neq \kappa$ let $\underset{\sim}{\eta}_i$ be the P_{i+1}-name of the $\underset{\sim}{Q}_i$-generic real

$$\underset{\sim}{\eta}_i = \cup\{tr(p(i)) : p \in \underset{\sim}{G}_{P_{i+1}}\}.$$

Then for $i < j$ from $< \mu$ of cofinality $\neq \kappa$ we have

$$\Vdash_{P_j} \text{“Rang}(\underset{\sim}{\eta}_i) \in \underset{\sim}{D}_j\text{”}$$

(*f*) $\underset{\sim}{\tau}_i$ is a P_i-name of a function from $\underset{\sim}{Q}_i$ to $\{h : h$ a function from a finite set of ordinals to $\mathscr{H}(\omega)\}$, such that:
$\Vdash_{P_i}$ “if $p, q \in \underset{\sim}{Q}_i$ are compatible <u>then</u> they have a common upper bound r such that $\underset{\sim}{\tau}_i(r) = \underset{\sim}{\tau}_i(p) \cup \underset{\sim}{\tau}_i(q)$”

(*g*) if $\mathrm{cf}(i) \neq \mu$ and $i \in \mathrm{Dom}(p), p \in P_j$ and $i < j \leq \mu$ then $\underset{\sim}{\tau}_i(p(i))$ is $\{\langle 0, tr(p)\rangle\}$; i.e. this is forced to hold

(*h*) we stipulate $P_i = \{p : p$ is a function with domain a finite subset of i for each $j \in \mathrm{Dom}(p), \emptyset_{P_j}$ forces that $p(j) \in \underset{\sim}{Q}_j$ and it forces a value to $\underset{\sim}{\tau}_j(p(j))\}$

(*i*) $\Vdash_{P_i}$ “$\underset{\sim}{Q}_i \subseteq \mathscr{H}_{<\aleph_0}(\gamma)$ for some ordinal γ”.

Let $\gamma(\mathfrak{t})$ be the minimal ordinal γ such that $i < \mu \Rightarrow \Vdash_{P_i}$ “if $x \in \underset{\sim}{Q}_i$ then $\mathrm{dom}(\underset{\sim}{\tau}_i(x)) \subseteq \gamma$”.
We let $\tau^{\mathfrak{t},i}$ be the function with domain P_i such that $\tau^{\mathfrak{t},i}(p)$ is a function with domain $\{\gamma(\mathfrak{t})j + \beta : j \in \mathrm{Dom}(p)$ and $p \restriction j \Vdash_{P_j}$ “$\beta \in \mathrm{Dom}(\underset{\sim}{\tau}_j(p(j))$”$\}$ and let $\tau^{\mathfrak{t},i}(\gamma(\mathfrak{t})j + \beta)$ be the value which $p \restriction j$ forces on $\underset{\sim}{\tau}^{\mathfrak{t}}_j(\beta)$.

Obviously
<u>4.4 Subclaim</u>: $\mathfrak{K} \neq \emptyset$.

Proof. Should be clear.

<u>4.5 Subclaim</u>: if in a universe $\mathbf{V}, D$ is a nonprincipal ultrafilter on ω then

(*a*) $\Vdash_{Q(\bar{D})}$ “$\{tr(p)(\ell) : \ell < \ell g(tr(p)), p \in \underset{\sim}{Q}_{Q(D)}\}$ is an infinite subset of ω, almost included in every member of D”

(*b*) $Q(D)$ is a c.c.c. forcing notion, even σ-centered

(*c*) $\underset{\sim}{\eta}_i = \cup\{tr(p) : p \in \underset{\sim}{G}_{Q(D)}\} \in {}^\omega\omega$ is forced to dominate $({}^\omega\omega)^{\mathbf{V}}$.

[Note that this, in particular clause (c), does not depend on additional properties of D; but as we naturally add many Cohen reals (by the nature of the support) we may add more and then can demand e.g. $\underset{\sim}{D}_i$ ($\mathrm{cf}(i) \neq \kappa$) is a Ramsey ultrafilter.]

4.6 Definition. 1) We define $\leq_{\mathfrak{K}}$ by: $\mathfrak{t} \leq_{\mathfrak{K}} \mathfrak{s}$ if ($\mathfrak{t}, \mathfrak{s} \in \mathfrak{K}$ and) $i \leq \mu \Rightarrow P^{\mathfrak{t}}_i \lessdot P^{\mathfrak{s}}_i$ and $i < \mu$ & $\mathrm{cf}(i) \neq \kappa \Rightarrow \Vdash_{P^{\mathfrak{s}}_i}$ "$\underset{\sim}{D}^{\mathfrak{t}}_i \subseteq \underset{\sim}{D}^{\mathfrak{s}}_i$" and $i < \mu \Rightarrow \Vdash_{P^{\mathfrak{s}}_i}$ "$\underset{\sim}{\tau}^{\mathfrak{t}}_i \subseteq \underset{\sim}{\tau}^{\mathfrak{s}}_i$".

2) We say $\mathfrak{t}$ is a canonical $\leq_{\mathfrak{K}}$-ub of $\langle \mathfrak{t}_\alpha : \alpha < \delta \rangle$ if:

(i) $\mathfrak{t}, \mathfrak{t}_\alpha \in \mathfrak{K}$

(ii) $\alpha \leq \beta < \delta \Rightarrow \mathfrak{t}_\alpha \leq_{\mathfrak{K}} \mathfrak{t}_\beta \leq_{\mathfrak{K}} \mathfrak{t}$

(iii) if $i < \mu$ and $\mathrm{cf}(i) = \kappa$ then $\Vdash_{P^{\mathfrak{t}}_i}$ "$\underset{\sim}{Q}^{\mathfrak{t}}_i = \bigcup\limits_{\alpha<\delta} \underset{\sim}{Q}^{\mathfrak{t}_\alpha}_i$".

Note that if $\mathrm{cf}(\delta) > \aleph_0$ then $\Vdash_{P^{\mathfrak{t}}_i}$ "$\underset{\sim}{Q}^{\mathfrak{t}}_i = \bigcup\limits_{\alpha<\delta} \underset{\sim}{Q}^{\mathfrak{t}_\alpha}_i$" for every $i < \mu$, so $\mathfrak{t}$ is totally determined.

3) We say $\langle \mathfrak{t}_\alpha : \alpha < \alpha^* \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous if: $\alpha < \beta < \alpha^* \Rightarrow \mathfrak{t}_\alpha \leq_{\mathfrak{K}} t_\beta$ and for limit $\delta < \alpha^*, \mathfrak{t}_\delta$ is a canonical $\leq_{\mathfrak{K}}$-ub of $\langle \mathfrak{t}_\alpha : \alpha < \delta \rangle$. Note that we have not say "the canonical $\leq_{\mathfrak{K}}$-u.b." as for $\delta < \alpha^*$, $\mathrm{cf}(\delta) \neq \kappa$ we have some freedom in completing $\cup \{\underset{\sim}{D}^{\mathfrak{t}_\alpha} : \alpha < \delta\}$ to an ultrafilter (on ω in $\mathbf{V}^{P^{\mathfrak{t}}_i}$).

<u>4.7 Subclaim</u>: If $P_1 \lessdot P_2$ and $\underset{\sim}{D}_\ell$ is a P_ℓ-name of a nonprincipal ultrafilter on ω for $\ell = 1, 2$ and $\Vdash_{P_2}$ "$\underset{\sim}{D}_1 \subseteq \underset{\sim}{D}_2$", <u>then</u> $P_1 * Q(\underset{\sim}{D}_1) \lessdot P_2 * Q(\underset{\sim}{D}_2)$.

[Why? First, we can first force with P_1, so without loss of generality P_1 is trivial and $D_1 \in \mathbf{V}$ is a nonprincipal ultrafilter on ω. Now clearly $p \in Q(D_1) \Rightarrow p \in Q(\underset{\sim}{D}_2)$ and $Q(D_1) \models p \leq q \Rightarrow Q(\underset{\sim}{D}_2) \models p \leq q$ and if $p, q \in Q(D_1)$ are incompatible in $Q(D_1)$ then they are incompatible in $Q(\underset{\sim}{D}_2)$. Lastly, in $\mathbf{V}$, let $\mathscr{I} = \{p_n : n < \omega\} \subseteq Q(D_1)$ be predense in $Q(D_1)$, we shall prove that $\mathscr{I}$ is predense in $Q(\underset{\sim}{D}_2)$. For this it suffices to note

$\boxtimes$ if D_1 is a nonprincipal ultrafilter on $\omega, \mathscr{I} \subseteq Q(D_1)$ and $\eta \in {}^{\omega>}\omega$, <u>then</u> the following conditions are equivalent

$(a)_\eta$ there is no $p \in Q(D_1)$ incompatible with every $q \in \mathscr{I}$ which satisfies $\mathrm{tr}(p) = \eta$

$(b)_\eta$ there is a set T such that:

(i) $\nu \in T \Rightarrow \eta \trianglelefteq \nu \in p$

(ii) $\eta \trianglelefteq \nu \trianglelefteq \rho \in T \Rightarrow \nu \in T$

(iii) if $\nu \in T$ then either $\{n : \nu\hat{\ }\langle n\rangle \in T\} \in D_1$ or $(\forall n)(\nu\hat{\ }\langle n\rangle \notin T)$ & $(\exists q \in \mathscr{I})(\nu = tr(q))$

(iv) there is a strictly decreasing function $h : T \to \omega_1$

(v) $\eta \in p$.

Proof of $\boxtimes$. Straight.

So as in $\mathbf{V}, \mathscr{I} \subseteq Q(D_1)$ is predense, for every $\eta \in {}^{\omega>}\omega$ we have $(a)_\eta$ for D_1 hence by $\boxtimes$ we have also $\eta \in {}^{\omega>}\omega \Rightarrow (b)_\eta$, but clearly if T_η witness $(b)_\eta$ in $\mathbf{V}$ for D_1, it witnesses $(b)_\eta$ in $\mathbf{V}^{P_2}$ for D_2 hence applying $\boxtimes$ again we get: $\eta \in {}^{\omega>}\omega \Rightarrow (a)_\eta$ in $\mathbf{V}^{P_2}$ for D_2, hence $\mathscr{I}$ is predense in $Q(D_2)$ in $\mathbf{V}^{P_2}$. So we have proved Subclaim 4.7.

<u>4.8 Subclaim</u>: If $\langle \mathfrak{t}_\alpha : \alpha < \delta\rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous and $\delta < \lambda^+$ is a limit ordinal, <u>then</u> it has a canonical $\leq_{\mathfrak{K}}$-ub.
[Why? By induction on $i < \mu$, we define $P^{\mathfrak{t}}_i$ and then $\underset{\sim}{Q}^{\mathfrak{t}}_i, \underset{\sim}{\tau}_i$ and $\underset{\sim}{D}_i$ (if $\mathrm{cf}(i) \neq \kappa$) such that the relevant demands (for $\mathfrak{t} \in \mathfrak{K}$ and for being canonical $\leq_{\mathfrak{K}}$-ub of $\bar{\mathfrak{t}}$) hold. Defining $P^{\mathfrak{t}}_i$ is obvious: for $i = 0$ trivially, if $i = j+1$ it is $P^{\mathfrak{t}}_j * Q^{\mathfrak{t}}_j$ and i is limit it is $\cup\{P^{\mathfrak{t}}_j : j < i\}$. As we have proved the relevant demands on $P^{\mathfrak{t}}_j, \underset{\sim}{Q}^{\mathfrak{t}}_j$ for $j < i$ clearly $P^{\mathfrak{t}}_i$ is c.c.c. by using $\langle \underset{\sim}{\tau}_j : j < i\rangle$ and clearly $\langle P^{\mathfrak{t}}_\zeta, \underset{\sim}{Q}^{\mathfrak{t}}_\xi : \zeta \leq i, \xi < i\rangle$ is an FS iteration. Now we shall prove that $\alpha < \delta \Rightarrow P^{\mathfrak{t}_\alpha}_i \lessdot P^{\mathfrak{t}}_i$?
So let $\mathscr{I}$ be a predense subset of $P^{\mathfrak{t}_\alpha}_i$ and $p \in P^{\mathfrak{t}}_i$ and we should prove that p is compatible with some $q \in \mathscr{I}$ in $P^{\mathfrak{t}}_i$; we divide the proof to cases.

<u>Case 1</u>: i is a limit ordinal.

So $p \in P^{\mathfrak{t}}_j$ for some $j < i$, let $\mathscr{I}' = \{q \restriction j : q \in \mathscr{I}\}$, so clearly $\mathscr{I}'$ is a predense subset of $P^{\mathfrak{t}_\alpha}_j$ (as $\mathfrak{t}_\alpha \in \mathfrak{K}$). By the induction hypothesis, in $P^{\mathfrak{t}}_j$ the condition p is compatible with some $q' \in \mathscr{I}'$; so let $r' \in P^{\mathfrak{t}}_j$ be a common upper bound of q', p and let $q' = q \restriction j$ where $q \in \mathscr{I}$. So $r \cup (q \restriction [j, i)) \in P^{\mathfrak{t}}_i$ is a common upper bound of q, p as required.

<u>Case 2</u>: $i = j+1$, $\mathrm{cf}(j) = \kappa$.

So without loss of generality for some $\beta < \delta, p(j)$ is a $P^{\mathfrak{t}_\beta}_j$-name of a member of $\underset{\sim}{Q}^{\mathfrak{t}_\beta}_j$; and without loss of generality $\alpha \leq \beta < \delta$. By the induction hypothesis $P^{\mathfrak{t}_\beta}_j \lessdot P^{\mathfrak{t}}_j$ hence there is $p' \in P^{\mathfrak{t}_\beta}_j$ such that $[p' \leq p'' \in P^{\mathfrak{t}_\beta}_j \Rightarrow p'', p \restriction j$ are compatible in $P^{\mathfrak{t}}_j]$.
Let

$$\mathscr{J} = \{q' \restriction j : q' \in P_i^{\mathfrak{t}_\beta} \text{ and } q' \text{ is above some member of } \mathscr{I}$$
$$\text{and } q' \restriction j \Vdash_{P_j^{\mathfrak{t}_\beta}} \text{“}p(j) \leq^{\underset{\sim}{Q}_j^{\mathfrak{t}_\beta}} q'(j)\text{”}\}.$$

Now $\mathscr{J}$ is a dense subset of $P_j^{\mathfrak{t}_\beta}$ (since if $q \in P_j^{\mathfrak{t}_\beta}$ then $q \cup \{\langle j, p(j)\rangle\}$ belongs to $P_i^{\mathfrak{t}_\beta}$ hence is compatible with some member of $\mathscr{I}$).
Hence p' is compatible with some $q'' \in \mathscr{J}$, so there is r such that $p' \leq r \in P_j^{\mathfrak{t}_\beta}, q'' \leq r$. As $q'' \in \mathscr{J}$ there is $q' \in P_i^{\mathfrak{t}_\beta}$ such that $q' \restriction j = q'', q'$ is above some $q^* \in \mathscr{I}$ and $q' \restriction j \Vdash$ “$p(j) \leq^{\underset{\sim}{Q}_j^{\mathfrak{t}_\beta}} q'(j)$”.

As $P_j^{\mathfrak{t}_\beta} \models$ “$p' \leq r \quad \& \quad q' \restriction j = q'' \leq r$” and by the choice of p' there is $p^* \in P_j^{\mathfrak{t}}$ above r (hence above p' and above $q'' = q' \restriction j$), and above $p \restriction j$. Now let $r^* = p^* \cup (q'' \restriction \{j\})$, clearly $r^* \in P_i^{\mathfrak{t}}$ is above $p \restriction j$ and $r^* \restriction j$ forces that $r^*(j)$ is above $p \restriction \{j\}$. Clearly $r^* \restriction j$ is above r and r^* also is above $q^* \in \mathscr{I}$ so we are done.

<u>Case 3</u>: $i = j+1$, $\mathrm{cf}(j) \neq \kappa$.

Use Subclaim C above.
So we have dealt with $\alpha < \delta \Rightarrow P_i^{\mathfrak{t}_\alpha} \lessdot P_i^{\mathfrak{t}}$.

If $P_i^{\mathfrak{t}}$ has been defined and $\mathrm{cf}(i) = \kappa$ we let $\underset{\sim}{Q}_i^{\mathfrak{t}} = \bigcup\limits_{\alpha<\delta} Q_i^{\mathfrak{t}_\alpha}$ and $\underset{\sim}{\tau}_i^{\mathfrak{t}} = \bigcup\limits_{\alpha<\delta} \underset{\sim}{\tau}_i^{\mathfrak{t}_\alpha}$, easy to check that they are as required. If P_i^t has been defined and $\mathrm{cf}(i) \neq \kappa$, then $\bigcup\limits_{\alpha<\delta} D_i^t$ is a filter on ω containing the co-bounded subsets, and we complete it to an ultrafilter. Clearly we are done.]

<u>4.9 Subclaim</u>: If $\mathfrak{t} \in \mathfrak{K}$ and E is a κ-complete nonprincipal ultrafilter on κ, <u>then</u> we can find $\mathfrak{s}$ such that:

- (i) $\mathfrak{t} \leq_{\mathfrak{K}} \mathfrak{s} \in \mathfrak{K}$
- (ii) there is $\langle \mathbf{k}_i, \mathbf{j}_i : i < \mu, \mathrm{cf}(i) \neq \kappa\rangle$ such that:
  - (α) $\mathbf{k}_i$ is an isomorphism from $(P_i^{\mathfrak{t}})^\kappa/E$ onto $P_i^{\mathfrak{s}}$
  - (β) $\mathbf{j}_i$ is the canonical embedding of $P_i^{\mathfrak{t}}$ into $(P_i^{\mathfrak{t}})^\kappa/E$
  - (γ) $\mathbf{k}_i \circ \mathbf{j}_i =$ identity on $P_i^{\mathfrak{t}}$
- (iii) $\underset{\sim}{D}_i^{\mathfrak{s}}$ is the image of $(\underset{\sim}{D}_i)^\kappa/E$ under $\mathbf{k}_i$ and similarly $\underset{\sim}{\tau}_i$ <u>if</u> $i < \mu$, $\mathrm{cf}(i) \neq \kappa$
- (iv) if $i < \mu$, $\mathrm{cf}(i) = \kappa$, then $\underset{\sim}{\tau}_i$ is defined such that, for $j < \kappa$, $\mathrm{cf}(j) \neq \kappa$ we have $\mathbf{k}_j$ is an isomorphism from $(P_i^{\mathfrak{t}}, \gamma', \tau^{\mathfrak{t},i})^\kappa/D$ onto $(P_i^{\mathfrak{s}}, \gamma'', \tau^{\mathfrak{s},i})$ for some ordinals γ', γ''.

[Why? Straight. Note that if $\mathrm{cf}(i) = \kappa, i < \mu$ then $\underset{\sim}{Q}^{\mathfrak{s}}_i$ is isomorphic to $P^{\mathfrak{s}}_{i+1}/P^{\mathfrak{s}}_i$ which is c.c.c. as by Łoś theorem for the logic $L_{\kappa,\kappa}$ we have $\bigcup\limits_{j<i}(P^t_j)^\kappa/E \lessdot (P^t_{i+1})^\kappa/E$, similarly for $\underset{\sim}{\tau}_i$ which guarantees that the quotient is c.c.c., too (actually $\underset{\sim}{\tau}_i$ is not needed for the c.c.c. here).]

<u>4.10 Subclaim</u> If $\mathfrak{t} \in \mathfrak{K}$ then $\Vdash_{P^{\mathfrak{t}}_\mu}$ "$\mathfrak{u} = \mathfrak{b} = \mathfrak{d} = \mu$".

[Why? In $\mathbf{V}^{P^{\mathfrak{t}}_\mu}$, the family $\mathscr{D} = \{\mathrm{Rang}(\underset{\sim}{\eta}_i) : i < \mu, \mathrm{cf}(i) \neq \kappa\} \cup \{[n,\omega) : n < \omega\}$ generates a filter on $\mathscr{P}(\omega)^{\mathbf{V}[P^{\mathfrak{t}}_\mu]}$, as $\mathrm{Rang}(\underset{\sim}{\eta}_i) \in [\omega]^{\aleph_0}$,
$i < j < \mu \ \ \& \ \ \mathrm{cf}(i) \neq \kappa \ \ \& \ \ \mathrm{cf}(j) \neq \kappa \Rightarrow \ \mathrm{Rang}(\underset{\sim}{\eta}_j) \subseteq^* \ \mathrm{Rang}(\underset{\sim}{\eta}_i)$.
Also it is an ultrafilter as $\mathscr{P}(\omega)^{\mathbf{V}[P^{\mathfrak{t}}_\mu]} = \bigcup\limits_{i<\mu} \mathscr{P}(\omega)^{\mathbf{V}[P^{\mathfrak{t}}_i]}$ and if $i < \mu$, then Rang $(\underset{\sim}{\eta}_{i+1})$ induces an ultrafilter on $\mathscr{P}(\omega)^{\mathbf{V}[P^{\mathfrak{t}}_{i+1}]}$. So $\mathfrak{u} \leq \mu$. Also $({}^\omega\omega)^{\mathbf{V}[P^{\mathfrak{t}}_\mu]} = \bigcup\limits_{i<\mu} ({}^\omega\omega)^{\mathbf{V}[P^{\mathfrak{t}}_i]}$, $({}^\omega\omega)^{\mathbf{V}[P^{\mathfrak{t}}_i]}$ is increasing with i and if $\mathrm{cf}(i) \neq \kappa$ then $\underset{\sim}{\eta}_i \in {}^\omega\omega$ dominate $({}^\omega\omega)^{\mathbf{V}[P^{\mathfrak{t}}_i]}$ by Subclaim 4.5, so $\mathfrak{b} = \mathfrak{d} = \mu$ as in previous cases. Lastly, always $\mathfrak{u} \geq \mathfrak{b}$ hence $\mathfrak{u} = \mu$.]

Now we define $\mathfrak{t}_\alpha \in \mathfrak{K}$ for $\alpha \leq \lambda$ by induction on α satisfying $\langle \mathfrak{t}_\alpha : \alpha \leq \lambda \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous such that $\mathfrak{t}_{\alpha+1}$ is gotten from $\mathfrak{t}_\alpha$ as in Subclaim 4.9. Let $P = P^{\mathfrak{t}_\lambda}_\mu$, so $|P| \leq \lambda$ hence $(2^{\aleph_0})^{\mathbf{V}^P} \leq (\lambda^{\aleph_0})^{\mathbf{V}}$ and easily equality holds.

We finish by
<u>4.11 Subclaim</u>: $\Vdash_{P_\lambda}$ "$\mathfrak{a} \geq \ \mathrm{cf}(\lambda)$".
[Why? Assume toward a contradiction that $\theta < \ \mathrm{cf}(\lambda)$ and $p \in P$ and $p \Vdash_P$ "$\underset{\sim}{\mathscr{A}} = \{\underset{\sim}{A}_i : i < \theta\}$ is a MAD family; i.e.

(i) $A_i \in [\omega]^{\aleph_0}$
(ii) $i \neq j \Rightarrow |\underset{\sim}{A}_i \cap \underset{\sim}{A}_j| < \aleph_0$
(iii) under (i) + (ii), $\underset{\sim}{\mathscr{A}}$ is maximal".

Without loss of generality $\Vdash_P$ "$\underset{\sim}{A}_i \in [\omega]^{\aleph_0}$". As $\mathfrak{a} \geq \mathfrak{b} = \mu$ by Subclaim F, we have $\theta \geq \mu$. For each $i < \theta$ and $m < \omega$ there is a maximal antichain $\langle p_{i,m,n} : n < \omega \rangle$ of P and there is a sequence $\langle \mathbf{t}_{i,m,n} : n < \omega \rangle$ of truth values such that $p_{i,m,n} \Vdash$ "$m \in$

$\underset{\sim}{A}_i \equiv \mathbf{t}_{i,m,n}$". We can find countable $w_i \subseteq \mu$ such that $\bigcup\limits_{m,n<\omega} \mathrm{Dom}(p_{i,m,n}) \subseteq w_i$. Possibly increasing w_i retaining countability, we can find $\langle R_{i,\gamma} : \gamma \in w_i\rangle$ such that

- (α) w_i has a maximal element and $\gamma \in w_i \backslash \{\max(w_i)\} \Rightarrow \gamma + 1 \in w_i$
- (β) $R_{i,\gamma}$ is a countable subset of $P^{\mathfrak{t}_\lambda}_\gamma$ and $q \in R_{i,\gamma} \Rightarrow \mathrm{Dom}(q) \subseteq w_i$
- (γ) for $\gamma_1 < \gamma_2$ in $w_i, q \in R_{i,\gamma_2} \Rightarrow q \restriction \gamma_1 \in R_{i,\gamma_1}$
- (δ) for $\gamma_1 \in w_i, \gamma \in \delta_1 \cap w_i$ and $q \in R_{i,\gamma_1}$ the $P^{\mathfrak{t}}_\gamma$-name $q(\gamma)$ involves $\aleph_0$ maximal antichains all included in $R_{i,\gamma}$
- (ε) $\{p_{i,m,m} : m, n\} \subseteq R_{i,\max(w_i)}$.

As $\mathrm{cf}(\lambda) > \aleph_0$ (otherwise the conclusion is trivial) we have $P^{\mathfrak{t}}_\mu = \bigcup\limits_{\alpha<\lambda} P^{\mathfrak{t}_\alpha}_\mu$. Clearly for some $\alpha < \lambda$ we have $\cup\{R_{i,\gamma} : i < \theta, \gamma \in w_i\} \subseteq P^{\mathfrak{t}_\alpha}_\mu$. But $P^{\mathfrak{t}_\alpha}_\mu \lessdot P^{\mathfrak{t}_\lambda}_\mu$. So $\Vdash_{P^{\mathfrak{t}_\alpha}_\mu}$ "$\underset{\sim}{\mathscr{A}} = \{\underset{\sim}{A}_i : i < \theta\}$ is MAD".

Now, letting $\mathbf{j}$ be the canonical elementary embedding of $\mathbf{V}$ into $\mathbf{V}^\kappa/D$, we know:

- $(*)$ in $\mathbf{V}^\kappa/D, \mathbf{j}(\underset{\sim}{\mathscr{A}})$ is a $\mathbf{j}(P^{\mathfrak{t}_\alpha}_\mu)$-name of a MAD family.

As $\mathbf{V}^\kappa/D$ is κ-closed, for c.c.c. forcing notions things are absolute enough but $\{\mathbf{j}(i) : i < \mu\}$ is not $\{i : \mathbf{V}^\kappa/D \models i < \mathbf{j}(\mu)\}$, so in $\mathbf{V}$, it is forced for $\Vdash_{\mathbf{j}(P^{\mathfrak{t}}_\mu)}$, that $\{\mathbf{j}(\underset{\sim}{A}_i) : i < \mu\}$ is not MAD!
Chasing arrows, clearly $\Vdash_{P^{\mathfrak{t}_{\alpha+1}}_\mu}$ "$\{\underset{\sim}{A}_i : i < \theta\}$ is not MAD" as required. $\square_{4.1}$

4.12 Theorem. *Assume*

- (a) κ *is a measurable cardinal*
- (b) $\kappa < \mu = \mathit{cf}(\mu) < \lambda = \mathit{cf}(\lambda) = \lambda^\kappa$.

Then for some c.c.c. forcing notion P of cardinality λ in $\mathbf{V}^P$ we have: $2^{\aleph_0} = \lambda, \mathfrak{i} = \mathfrak{u} = \mathfrak{d} = \mathfrak{b} = \mu$ and $\mathfrak{a} = \lambda$.

Remark. Recall $\mathfrak{i} = \mathrm{Min}\{|\mathscr{A}| : \mathscr{A} \subseteq [\omega]^{\aleph_0}$ is a maximal independent$\}$ where independent means that every nontrivial Boolean combination of finitely many members of $\mathscr{A}$ is not empty and even infinite.

Proof. Like the proof of 4.1 except for the following changes.

Let $S_0 = \{i < \mu : \mathrm{cf}(i) = \kappa\}, S_1 = \{i < \mu : \mathrm{cf}(i) \neq \kappa, i \text{ even}\}$, $S_2 = \{i < \mu : \mathrm{cf}(i) \neq \kappa, i \text{ odd}\}$, so $\langle S_0, S_1, S_2\rangle$ is a partition of μ. For $i \in S_0, i \in S_1$ we define

$\mathfrak{t} \in \mathfrak{K}$ as before (so in clauses (c), (e), we restrict ourselves to $i \in S_1$ and $j \in S_1$) but we add

(j) for $i \in S_2, \underset{\sim}{D}_i$ is a maximal filter on ω containing the co-bounded subsets such that $\{\text{Rang}(\underset{\sim}{\eta}_j) : j \in i \cap S_2\}$ is an independent family modulo $\underset{\sim}{D}_i$

(k) for $i \in S_2$ and $j \in i \cap S_2$ we have $\underset{\sim}{D}_j \subseteq \underset{\sim}{D}_i$.

In Subclaim A, define $\bar{Q}^{\mathfrak{t}} \restriction i$ by induction on i such that $\Vdash_{P_i}$ "$\langle \text{Rang}(\underset{\sim}{\eta}_j) : j \in S_2 \cap i\rangle$ is independent modulo $\cup\{\underset{\sim}{D}_j : j \in i \cap S_2\}$".
Note the definition of $Q(\underset{\sim}{D}_i)$ remains: $\text{tr}(p) \triangleleft \eta \in p \in Q(\underset{\sim}{D}_i) \Rightarrow \{n : \eta\hat{\ }\langle n\rangle \in p\}$ belongs to D_i. Also note

$\boxtimes_1$ "if $A \subseteq \omega, A \neq \emptyset$ mod $\underset{\sim}{D}$ then $\Vdash_{Q(D)}$ " $\text{Rang}(\underset{\sim}{\eta}) \cap A \neq \emptyset$ mod D.
However

$\boxtimes_2$ in Subclaim C we assume $\Vdash_{P_2}$ "$\underset{\sim}{D}_1 = \underset{\sim}{D}_2 \cap \mathscr{P}(\omega)^{\mathbf{V}^{P_1}}$" and in the proof we replace $\boxtimes(b)(iii)$ by:

$(iii)'$ if $\nu \in T$ then $\{n : \nu\hat{\ }\langle n\rangle \in T\} \neq \emptyset$ mod D or
$(\forall n)(\nu\hat{\ }\langle n\rangle \notin T)$ & $(\exists q \in \mathscr{I})(\nu \in q)$.

Also we should add
<u>4.13 Subclaim</u>: If $\mathfrak{t} \in \mathfrak{K}$ then $\Vdash_{P^{\mathfrak{t}}_\mu}$ "$\{\text{Rang}(\underset{\sim}{\eta}_j) : j \in S_2\}$ is a maximal independent family of subsets of ω".
[Why? Independence is covered by the previous paragraph. Assume towards a contradiction that $p \in P^{\mathfrak{t}}_\mu$ and $p \Vdash$ "$\underset{\sim}{A} \in [\omega]^{\aleph_0} \backslash \{\text{Rang}(\underset{\sim}{\eta}_j) : j \in S_2\}$ and $\{\text{Rang}(\underset{\sim}{\eta}_j) : j \in S_2\} \cup \{\underset{\sim}{A}\}$ is independent". For some $j \in S_2$ we have $p \in P^{\mathfrak{t}}_j$ and $\underset{\sim}{A}$ is a $P^{\mathfrak{t}}_j$-name. So in $\mathbf{V}^{P^{\mathfrak{t}}_j}, \{\text{Rang}(\underset{\sim}{\eta}_i) : i \in S_2 \cap j\} \cup \{\underset{\sim}{A}\}$ is an independent family of subsets of ω, so by the maximality of $\underset{\sim}{D}_j$ we have: for some $m < n < \omega, i_\ell \in S_2 \cap j$ for $\ell < n$ with no repetitions we have $\bigcap\limits_{\ell=0}^{m-1} \text{Rang}(\underset{\sim}{\eta}_{i_\ell}) \cap \bigcap\limits_{\ell=m}^{n-1} (\omega \backslash \text{Rang}(\underset{\sim}{\eta}_{i_\ell})) \cap \underset{\sim}{A} \cap B = 0$ for some $B \in D_j$ (maybe interchange A and its complement), so without loss of generality p forces this (with B replaced by a $P^{\mathfrak{t}}_j$-name B). But $\Vdash_{P^{\mathfrak{t}}_{j+1}}$ "$\text{Rang}(\underset{\sim}{\eta}_j) \cap (\omega \backslash \underset{\sim}{B})$ is finite" hence

$$p \Vdash \text{“} \bigcap_{\ell=0}^{m-1} \text{ Rang}(\underset{\sim}{\eta}_{i_\ell} \cap \bigcap_{\ell=m}^{n-1} (\omega \setminus \text{ Rang}(\underset{\sim}{\eta}_{i_\ell})) \cap \text{ Rang}(\underset{\sim}{\eta}_j) \cap A \text{ is finite”}.$$

This contradicts the choice of $\underset{\sim}{A}$ so we are done. $\square$

$\square_{4.12}$

4.14 Discussion: 1) We can now look at other problems, like what can be the order and equalities among $\mathfrak{d}, \mathfrak{b}, \mathfrak{a}, \mathfrak{u}, \mathfrak{i}$; have not considered it.

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